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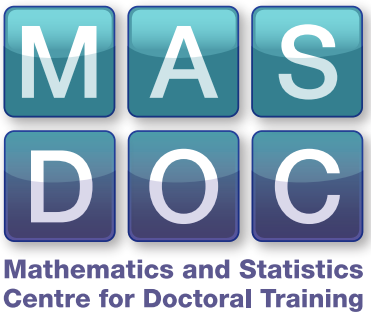

RSK Correspondence and Pitman's Theorem

by

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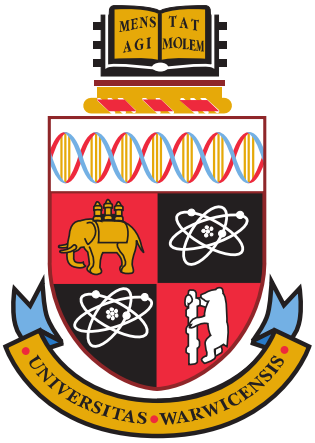

Contents

Acknowledgements **iii**

Declarations **iv**

Abstract **v**

1 Combinatorics **1**

1.1 Introduction 1

1.2 Young Diagrams and Tableaux 2

1.3 RSK Correspondence, Classical Definitions 3

1.4 RSK Algorithm, Alternative Definitions 8

1.5 Schur functions 11

2 RSK with Random Input **13**

2.1 Dynamics 14

3 Pitman's Theorem **17**

3.1 3-Dimensional Bessel Process 17

3.2 Brownian Motion with a Constant Drift 18

4 Scaling from RSK to Pitman **20**

4.1 The Plan 20

4.2 Choosing Parameters 21

4.3 The Intertwining Kernel 23

4.4 Any Luck in Generalisation? 27

5 Extensions of Pitman and RSK **29**

5.1 Multi-dimension 29

5.2 Geometric Lifting 34

5.3 Continuum 39

6 Applications **42**

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Declarations

I, Yuchen Pei, declare that to the best of my knowledge this work constitutes an original review compiled by the author except when otherwise specified by the references.

The material in this thesis has not to my knowledge been submitted for any other degree either at this university (the University of Warwick) or any other university.

Abstract

The largest eigenvalue of GUE (Gaussian Unitary Ensemble) and HBM (Hermition Brownian Motion) are path-functional and path-transform of Brownian motions, which are a scaling limit of random walks. The discrete transform is a directed first passage percolation model, which can be connected naturally with RSK (Robinson-Schensted-Knuth) correspondence. Meanwhile this model describes discrete directed random polymer with temperature 0. Changing the temperature to positive can lead to a geometric version of the path transform. These path transforms give rise to various versions of RSK correspondence and Pitman's theorems, including classical and new ones. This dissertation surveys this development and makes an attempt to generalise Pitman's theorem to Brownian motion with variable drift by inferring from the discrete model involving RSK correspondence.

Chapter 1

Combinatorics

1.1 Introduction

The story dates back to the study of Ulam's problem of distribution of $l_1(\sigma)$, the length of longest increasing subsequence of permutation σ chosen uniformly randomly in the late 1990s. $l_1(\sigma)$ is the length of the first row of Young's tableau obtained by performing Robinson-Schensted algorithm on σ. In the work [BDJ99], this connection is used to obtain the asymptotics of this distribution, and it turns out to the the Tracy-Widom law [TW94] that is the distribution of the largest eigenvalue of Gaussian Unitary Ensemble. Then in 1999, extending the result to Robinson-Schensted-Knuth correspondence, [Joh00] obtained similar asymptotics for directed last passage percolation and totally asymmetric simple exclusion processes. Meanwhile, the shape of tableaux obtained by RSK correspondence can be viewed as a path transform of the walks, either random or deterministic, which in dimension 2 concurs with Pitman's classical $2M - X$ theorem. Motivated by these connections and Burke's theorem for M/M/1 queues which also interpretes Pitman's theorem, around 2000 [OY01, OY02, O'C03a, O'C03b] found an extension for Burke's theorem, associated the path transforms with series of queues in series and generalised Pitman's theorem to multidimensional setting. About the same time Pitman's theorem was also considered in geometric setting by [MY00] and geometric RSK was developed by [Kir01, NY04]. However it was not until about 2010s were the system and connection of path-transform, RSK correspondence and multi-dimensional Pitman's theorem built up in geometric lifting and continuum contexts [COSZ11, OW11, BO11, O'C12], some of which were also motivated by the development of directed random polymer and KPZ equation and universality class.

The rest of this dissertation is organised as follows. In the rest of this chapter, an introduction to relevant combinatroics including RSK correspondence and Schur functions are given. In Chapter 2 we consider the RSK correspondence with random input and write

down its dynamics. In Chapter 3 the development of Pitman's $2M - X$ theorem is recorded, whose generalisation was attempted by using scaling limit of RSK in Chapter 4. In Cahpter 5 various versions of RSK correspondence and Pitman's theorem are surveyed. And the dissertation is closed in Chapter 6 with a brief summary of the applications.

1.2 Young Diagrams and Tableaux

Young diagrams and tableaux are combinatorial objects useful in group representation theory, for example they can be used for describing all irriducible representations of symmetric groups $\mathcal{S}_n$. For an introduction see e.g. [Sag00, Ful97, Sta01]. A *Young diagram* λ is a vector $\lambda = (\lambda_1, \dots, \lambda_k)$ for some $k \in \mathbb{N}$ and all $\lambda_i \in \mathbb{N}_+$ such that $\lambda_1 \geq \cdots \geq \lambda_k > 0$. It is visualised as an array of left adjusted boxes, with the ith row having λ_i boxes. By denoting $|\lambda|$ as sum of entries of λ, each diagram λ can also be regarded as a partition of $|\lambda|$. We denote the *length* of a diagram λ by $l(\lambda)$ as the number of rows in it. For example, in Figure 1.1 is a Young diagram $\lambda = (4, 3, 1, 1) \vdash 9$ with length $l(\lambda) = 4$.

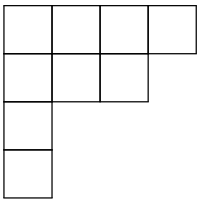

Figure 1.1: A Young diagram $\lambda = (4, 3, 1, 1)$ with 4 rows

A Young tableau P is a Young diagram λ filled with positive integer numbers. We call the associated Young diagram its *shape*, denoted as $\lambda = \mathrm{sh}P$. Similarly, the *type* of a tableau P, denoted as $\mathrm{ty}P$, is a composition of $|\mathrm{sh}P|$ whose i'th entry is the number of i's in P. A Young tableau P is called *standard* if the numbers are uniquely picked from $\{1, 2, \dots, |\mathrm{sh}P|\}$ and strictly increasing along columns and rows, called *semistandard* if numbers are strictly increasing along columns and weakly increasing along rows. We denote by $\mathcal{T}_l$ and $\mathcal{T}_l^s$ as the set of semistandard and standard tableaux with entries in $[l]$ respectively. Also denote $\mathcal{T} := \bigcup_{l \geq 1} \mathcal{T}_l$ as the set of all semistandard tableaux. In Figure 1.2 are a tableau, a standard tableau, a semistandard tableau with shape $(4, 3, 1, 1)$. And for example, the type of the first tableau is $(0, 0, 2, 2, 1, 0, 2, 1, 1, 0, 0, 0, \dots)$. From next section on we always restrict the possibility of numbers appearing in tableaux to $[n]$ or $[k]$, hence we can get rid of the infinite 0's in the type of tableaux by restricting them on $\mathbb{N}^n$ and $\mathbb{N}^k$ respectively.

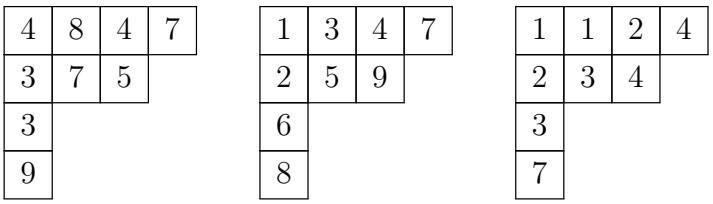

Figure 1.2: From left to right: a Young tableau, a standard Young tableau and a semistandard Young tableau

1.3 RSK Correspondence, Classical Definitions

For any $\lambda \vdash n$ denote f^λ to be the number of standard tableaux with shape λ. It is the multiplicity and dimension of Specht module S^λ in regular representation of symmetric group $\mathcal{S}_n$ (see e.g. [Sag00]). Therefore we have

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!. \tag{1.1}$$

By dividing both side of this identity by $n!$ we obtain the *Plancherel measure* of $\mathcal{S}_n$ on the set of all partitions / Young diagrams:

$$P_{\mathrm{pl}}(\lambda) = \frac{(f^\lambda)^2}{|\mathcal{S}_n|}.$$

The algebraic result (1.1) is proved in a totally combinatorial way by constructing a one-one correspondence between $\mathcal{S}_n$ and set of pairs of standard tableaux with the same shape.

$$\sigma \leftrightarrow (P, Q).$$

Such a correspondence was defined independently by Robinson and Schensted [Rob38, Sch61], called Robinson-Schensted correspondence. It was later generalised by Knuth [Knu70]. For clarity, we first describe the generalisation, i.e. Robinson-Schensted-Knuth algorithm for generalised permutations.

Fix n and k as positive integers. A *generalised permutation* π is a pair of vectors $(\hat{\pi}, \check{\pi})^T \in ([n] \times [k])^N$ in lexicographical order, i.e.

$$\pi = \begin{pmatrix} \hat{\pi}_1 & \dots & \hat{\pi}_N \\ \check{\pi}_1 & \dots & \check{\pi}_N \end{pmatrix}, \quad (\hat{\pi}_i, \check{\pi}_i) \in [n] \times [k] \forall i$$

satisfying

$$\text{Either } \hat{\pi}_i < \hat{\pi}_{i+1} \text{ Or if } \hat{\pi}_i = \hat{\pi}_{i+1} \text{ then } \check{\pi}_i \leq \check{\pi}_{i+1}, \quad i = 1, \ldots, N-1.$$

We denote by GP the set of all generalised permutations. The upper row $\hat{\pi}$ can be seen as an "index" and the lower the "content" of the generalised permutation, as will be seen from the algorithm.

The algorithm reads the paired entries one by one and update the output by proceeding row insertion to get paired semistandard tableaux $((P(i), Q(i)) : i = 1, \ldots, N)$ with $\mathrm{sh}P(i) = \mathrm{sh}Q(i), \forall i$, as follows:

1. Initialise $P(0) = Q(0) = \emptyset$; set $i = 1$.
2. If $i > N$ then the algorithm is done.
3. Set $j \leftarrow 1$, $P(i) \leftarrow P(i-1)$, $Q(i) \leftarrow Q(i-1)$.
4. Let $\check{\pi}_i$ displace the most west number s in jth row bigger than $\check{\pi}_i$ and let $\check{\pi}_i \leftarrow s$. If s does not exist then append $\check{\pi}_i$ at the end of the row and let $\check{\pi}_i \leftarrow \infty$.
5. If $\check{\pi}_i \neq \infty$ then set $j \leftarrow j + 1$ and goto step 4; otherwise, append a box to $Q(i)$ so that $\mathrm{sh}P(i) = \mathrm{sh}Q(i)$, fill the box with $\hat{\pi}_i$, set $i \leftarrow i + 1$ and goto step 2.

The two output tableaux $P(i)$ and $Q(i)$ are called *insertion tableau* (or P-tableau) and *recording tableau* (or Q-tableau) respectively. In the algorithm, Step 4 is called a *row insertion*, as it insert a number into a row of P-tableau. We denote by $P(\pi) := P(N)$ and $Q(\pi) := Q(N)$ the resultant tableaux. Clearly the P-tableau only depends on $\check{\pi}$, therefore we can also write $P(\check{\pi}) := P(\pi)$. For example, set $n = 4$ and $k = 3$, in Figure 1.3 are P- and Q- tableaux of generalised permutation

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \end{pmatrix}. \tag{1.2}$$

The RSK algorithm can take various inputs and give different versions of output. For example, a generalised permutation π can be indentified with a $n \times k$ matrix $D = (\xi_{ij})$ such that

$$\xi_{ij} = \#\{l \in [N] : (\hat{\pi}_l, \check{\pi}_l) = (i, j)\};$$

$P(\pi) =$

1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	3
2	2	2	3	3											
3	3														

$Q(\pi) =$

1	1	1	1	1	1	1	1	2	3	3	3	3	3	4	4
2	2	2	2	3											
3	3														

Figure 1.3: The output tableaux of π in (1.2).

in this case we write $D(\pi) := D$. Conversely each non-negative integer matrix $D = (\xi_{ij}) \in M(\mathbb{N})_{n\times k}$ can be turned into a generalised permutation

$$\begin{pmatrix} 1^{d_{11}} & \dots & 1^{d_{1k}} & \dots & i^{d_{ij}} & \dots & n^{d_{n1}} & \dots & n^{d_{nk}} \\ 1^{d_{11}} & \dots & k^{d_{1k}} & \dots & j^{d_{ij}} & \dots & 1^{d_{n1}} & \dots & k^{d_{nk}} \end{pmatrix}$$

where i^l stands for $\underbrace{i \quad i \quad \dots i}_{l\ i'\mathrm{s}}$. We can write D's associated generalised permutation as $\pi(D)$. For example, the permutation π in (1.2) is identified with 4×3 matrix D:

$$\pi \leftrightarrow D = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Thus the RSK algorithm can take a $n \times k$ matrix as an input instead, and we write $(P(D), Q(D)) := (P(\pi(D)), Q(\pi(D)))$. We will treat the row dimension of D as time, and due to the fact that the lower rows in D corresponds to righter pairs in π, it is better that we denote $P(m)$ and $Q(m)$ as the output after operating on all the pairs corresponding to the first m rows in D.

RSK algorithm is a bijective map between $M(\mathbb{N})_{n\times k}$ and $\{(P, Q) \in \mathcal{T}_k \times \mathcal{T}_n : \mathrm{sh}P = \mathrm{sh}Q\}$. This is why it's called a correspondence.

Example 1.1 (Original RS correspondence and permutations)**.** For any permutation $\sigma \in \mathcal{S}_n$, its two-line form can be identified with a generalised permutation

$$\pi = \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}.$$

After RSK algorithm the resultant tableaux $P(\pi)$ and $Q(\pi)$ are both standard. This defines the original RS correspondence, i.e. a bijection between $\mathcal{S}_n$ and $\{(P,Q) \in \mathcal{T}_n^s \times \mathcal{T}_n^s : \mathrm{sh}P = \mathrm{sh}Q\}$, which gives the identity (1.1). Note that the matrix $D(\pi)$ is its permutation matrix, i.e. $D \in \{0,1\}^{n\times n}$ with $(D)_{ij} = \delta_{\sigma(i),j}$.

If we replace the words "row" with "column" and "west" with "north" in RSK algorithm we have RSK with *column insertion*. which ristricted on $\mathcal{S}_n$ is still well-defined. We can still obtain a pair of output standard tableaux. They are transpose of the tableaux produced from row insertion, hence a bijection again.

Take $n = 9$ and $\sigma = (286159347)$ for example, then the corresponding tableaux P and Q after RS algorithm are

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 6 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \quad Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 9 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

Example 1.2 (RS correspondence)**.** It should be noted that in some literature, e.g. [O'C03a] RS correspondence is defined by restricting RSK on $\{\pi \in \mathrm{GP} : \hat{\pi} = (1,\ldots,n)\}$. For any such π we call the vector $w := \check{\pi}$ a *word* from the *alphabet* $[k]$. The associated input matrices are now the ones with only one non-zero entry per row. The image of the map is $\{(P,Q) \in \mathcal{T}_k \times \mathcal{T}_n^s : \mathrm{sh}P = \mathrm{sh}Q\}$. and it's also a bijection.

As stated in Example 1.1, when entries of $\hat{\pi}$ and $\check{\pi}$ are distinct, the row and column insertions have no qualitative difference. However this can't be generalised to common input matrices, as a transpose of semistandard tableau does not remain semistandard necessarily. Therefore a *modified column insertion* is introduced. Such an operation is no different from column insertion except the numbers for inserting displace the most north entry *greater than or equal* to them. This makes insertion tableaux semistandard. However the recording tableaux are still not guaranteed semistandard. The solution is to restrict the domain of the algorithm to the set of matrices D with only 0 and 1 entries. And by doing so, all pairs of entries in the generalised permutation are distinct, resulting in the transpose of Q tableaux being semistandard. And once again it's a bijection, but between $\{0,1\}^{n\times k}$ and $\mathcal{T}_k \times \mathcal{T}_n^t$, where $\mathcal{T}_n^t$ is the set of tableaux whose transpose are in $\mathcal{T}_n$. This algorithm is called *dual RSK correspondence*, denoted as RSK'. Similarly the output of RSK' are denoted as $P'(\pi)$ and $Q'(\pi)$.

The dual RSK correspondence got its name for its dual-ish relationship with RSK. To see this, for any vector $a = (a_1, a_2, \ldots, a_n)$, denote $a^r = (a_n, a_{n-1}, \ldots, a_1)$ to be the reverse of a.

Because the row insertions and modified column insertions commute, the insertion tableau obtained from one algorithms is the same as those obtained from the other with an reversed input:

$$P(\check{\pi}) = P'(\check{\pi}^r). \tag{1.3}$$

Specifically, if we consider the case in Example 1.1, i.e. restricting on permutations, then since the modified column insertion is the same as normal column insertion which produces a transposed P-tableau, the reversed permutation produces the transpose of the P-tableau of the permutation.

$$P(\sigma^r) = P'(\sigma) = P(\sigma)^t. \tag{1.4}$$

The shape of the output tableaux is connected to the longest increasing subsequence. An *increasing* (or *decreasing*) *subsequence* of a generalised permutation π is a subsequence of pairs

$$\begin{pmatrix} \hat{\pi}_{i_1} & \dots & \hat{\pi}_{i_l} \\ \check{\pi}_{i_1} & \dots & \check{\pi}_{i_l} \end{pmatrix}$$

such that $(\check{\pi}_{i_1}, \dots, \check{\pi}_{i_l})$ is an increasing (or decreasing) subsequence of $\check{\pi}$. A *m-increasing* (or *decreasing*) subsequence of π is a subsequence composed of m disjoint increasing (or decreasing) subsequences. The definition of weakly increasing (or decreasing) subsequences are as expected. Denote $l_m(\pi)$ as the length of longest weakly m-increasing subsequence of π The result is due to Greene:

Theorem 1.3 (Greene's Theorem)**.** *Denote* $\lambda = shP(\pi)$*, then*

$$\lambda_1 + \cdots + \lambda_m = l_m(\pi).$$

By (1.3) the shape of RSK' output has a similar result for length of longest weakly m-decreasing subsequences, which in the case of original RS, is the transpose of the shape by (1.4). Also, when taking $m = 1$, this theorem states the length of longest weakly increasing subsequence is the same as the length of the first row of output tableaux of RSK.

1.4 RSK Algorithm, Alternative Definitions

In the previous section, we successfully reduce the input from the bulky repetitive generalised permutations to decent condensed matrices. It appears the output tableaux also have redundant information. In this section we represent the insertion and recording tableaux in a lighter way and give an alternative definition of RSK algorithm. As a side note, most of this section is inspired by the construction of tropical RSK correspondence (see e.g. [COSZ11]) and shouldn't be original. For $(x, y) \in \mathbb{R}^l \times \mathbb{R}^{l+1}$, define an interlacing relationship $\prec$ by

$$x \prec y \Leftrightarrow y_1 \geq x_1 \geq y_2 \geq x_2 \geq \cdots \geq x_l \geq y_{l+1}.$$

And define GT_l the set of l-level *Gelfand-Tsetlin cones* by

$$\mathrm{GT}_l = \{(x^1, \ldots, x^l) \in \mathbb{R}^1 \times \cdots \times \mathbb{R}^l : x^1 \prec \cdots \prec x^l\}.$$

Then it can be easily checked that the output tableaux $P(n)$ and $Q(n)$ at time n can be identified with Gelfand Tsetlin cones

$$\begin{aligned} P(n) &= (\mathrm{sh}P^1(n), \ldots, \mathrm{sh}P^k(n)) \in \mathrm{GT}_k; \\ Q(n) &= (\mathrm{sh}Q^1(n), \ldots, \mathrm{sh}Q^n(n)) \in \mathrm{GT}_n, \end{aligned} \tag{1.5}$$

where $P^j(n) \subset P(n)$ and $Q^j(n) \subset Q(n)$ stand for the largest subtableaux of $P(n)$ and $Q(n)$ containing numbers $1, \ldots, j$ in them (call them jth subtableaux of $P(n)$ and $Q(n)$ respectively). However there are quite a few 0 entries in both cones. To discard some redundant information, we deonte $L_i^j(m) := (\mathrm{sh}P^j(m))_i$ for $1 \leq i \leq j \leq k$ and $L^j(m) := (L_1^j, \ldots, L_j^j)(m)$. The previous correspondence (1.5) can be written as

$$P(n) = (L^1(n), L^2(n), \ldots, L^k(n)); \tag{1.6}$$

$$Q(n) = (L^k(1), L^k(2), \ldots, L^k(n)). \tag{1.7}$$

For example the output of π in (1.2) can be identified as

$$\begin{aligned} P(4) &= ((9), (15, 3), (16, 5, 2)); \\ Q(4) &= ((8, 0, 0), (9, 4, 0), (14, 5, 2), (16, 5, 2)). \end{aligned}$$

Note that $L_i^j(m) \neq 0$ only if $i \leq j \wedge m$. Also the fact that the shape of $P(n)$ and $Q(n)$ are the same is reflected by the same $L^k(n)$ as the last entries of them in (1.6) and (1.7). Therefore

the information of $P(n)$ and $Q(n)$ can be represented as

$$(P,Q)(n) = (L_i^j(m))_{(i,j,m)\in B_{n,k}},$$

where

$$B_{n,k} = \{(i,j,m) : 1 \le i \le j \wedge m, 1 \le j \le k, m = n\} \tag{1.8}$$
$$\sqcup\{(i,j,m) : 1 \le i \le j \wedge m, 1 \le m \le n-1, j = k\}. \tag{1.9}$$

The set $B_{n,k}$ has exactly nk elements. So we can identify the output with an n by k matrix $\tilde{D} = (\tilde{\xi}_{ij})$ with $P(n)$ occupying the northwest chunk and $Q(n)$ the southeast and sharing the border as their common shape.

$$\tilde{\xi}_{ij} = \begin{cases} L_i^{i+j-1}(n) & 1 \le i \le k \wedge n, 1 \le j \le k-i+1; \\ L_{k-j+1}^{k}(k+n-i-j+1) & 1 \le j \le k, k-j+1 \le i \le n. \end{cases}$$

For example, the output of (1.2) can be written as

$$(P,Q)(n) = \tilde{D} = \begin{pmatrix} 9 & 15 & 16 \\ 3 & 5 & 14 \\ 2 & 5 & 9 \\ 2 & 4 & 8 \end{pmatrix}$$

So we can define RSK correspondence as an onto map from $M(\mathbb{N})_{n\times k}$ to itself.

Now we can describe the RSK algorithm in a more compact way, which will be echoed in Section 5.2. Given two l-vectors a and b, a row insertion inserting b into a is a function $r : (\mathbb{N}^l \times \mathbb{N}^l) \to (\mathbb{N}^l \times \mathbb{N}^{l-1})$ defined by

$r : (a,b) \mapsto (a',b')$ s.t.

$a'_1 = b_1 + a_1; \qquad a'_i = b_i + (a'_{i-1} \vee a_i); \qquad b'_{i-1} = b_i + (a_i + a'_{i-1}) - (a_{i-1} + a'_i), \quad 2 \le i \le l.$

Denote $L_i(m) := (L_i^i, \ldots, L_i^k)(m)$ for $1 \le i \le k$ as the ith row of $P(m)$ and $\xi^m := (\xi_{m1}, \ldots, \xi_{mk})$ to be the mth row of input matrix . The RSK algorithm is re-described as below.

1. Initialise $P(0) = \emptyset$, i.e. $L_i^j(0) = 0, 1 \le i \le j \le k$. Set $m = 1$.

2. If $m > n$ then we are done.

3. Set i=1.

4. Insert ξ^m into $L_i(m-1)$ to update ξ^m and obtain $L_i(m)$: $(L_i(m), \xi^m) \leftarrow r(L_i(m-1), \xi^m)$.

5. If $i < m \wedge k$ then $i \leftarrow i+1$ and go to step 4; otherwise $m \leftarrow m+1$ and go to step 2.

Clearly, $(L_i^j(m))_{1\le i\le j\wedge m, 1\le j\le k, 1\le m\le n}$ covers all the information about both tableaux up to time n. Actually in the spirit of Greene's theorem, L has a very intuitive interpretation and nice structure.

For any $m\times j$ matrix A with non-negative integer entries, denote $T_l(A)$ to be the maximal sum of l non-intersecting directed paths in D from $(1,1),(1,2),\dots,(1,l)$ to $(j-l+1,m),(j-l+2,m),\dots,(j,m)$ respectively. More precisely for matrix $A\in M(\mathbb{N})_{m\times j}$,

$$T_l(A) = \max_{\substack{\pi_1:(1,1)\to(m,j-l+1),\pi_2:(1,2)\to(m,j-l+2),\dots,\pi_l:(1,l)\to(m,j)\\ \pi_1,\pi_2,\dots,\pi_l \text{ are non-intersecting}}} \sum_{(i,p)\in\pi_1\cup\cdots\cup\pi_l} (A)_{i,p}, \tag{1.10}$$

where $(i,j)\to(i',j')$ is a set

$$\begin{aligned}\pi := \{&(i_1,j_1),(i_2,j_2),\dots,(i_N,j_N):\\ &(i_1,j_1)=(i,j),(i_N,j_N)=(i',j'),(i_{p+1},j_{p+1})-(i_p,j_p)=(0,1) \text{ or } (1,0),\forall p\in[N]\},\end{aligned}$$

where $N = i'-i+j'-j+1$.

Then by Greene's theorem,L satisfies

$$L_1^j(m)+\cdots+L_l^j(m) = T_l(D^j(m)), \tag{1.11}$$

where $D^j(m) = (\xi_{i,p})_{m\times j}$ is a submatrix of D consisting its first m rows and j columns.

We close this section by giving one more form of input for RSK correspondence, the walks. For a $n\times k$ non-negative integer matrix D, define its associated k-dimensional walk $(S_1(m),\dots,S_k(m))_{0\le m\le n}$ on $\mathbb{N}^k$ by

$$S_j(0)=0; \quad S_j(m)=S_j(m-1)+\xi_{m,j}, \quad \forall j=1,\dots,k.$$

Clearly given such a walk, we can also recover the matrix D. Similarly we define $S^r = (S_k, S_{k-1},\dots,S_1)$ as a reversed (in index) walk. This kind of input will be used when we consider the shape of output tableaux as a path transform of the walks.

To sum up, we've defined RSK correspondence taking input as a generalised permutation π, a matrix D, a permutation σ, a word w or a walk S; and giving output as a pair of

tableaux (P, Q) with the same shape, $(L_i^j(m))$ the components of the tableaux or a matrix $\tilde{D}$.

1.5 Schur functions

Enumerative combinatorial objects tend to have many definitions. A notorious example is Catalan numbers, which according to [Sta01] can be interpreted in at least 66 ways. The all important Schur functions, a.k.a. Schur polynomials are slightly better, and we record four definitions appearing in many texts introducing these functions. Define for any tableau T $x^T = x^{\mathrm{ty}T}$ and for arbitrary sequence μ, x^μ is defined as

$$x^\mu := \prod_{i=1}^{\infty} x_i^{\mu_i}.$$

A Schur function associated with a partition $\lambda \vdash n$ is a symmetric function, thus a formal series and can take an arbitrary number of arguments.

$$s_\lambda(x) = \sum_{T \in \mathcal{T}:\mathrm{sh}T=\lambda} x^T = \sum_\mu K_{\lambda\mu} x^\mu$$

where the sum in the second identity is over composite μ of n, and $K_{\lambda\mu}$ are *Kostka numbers*, the multiplicity of Specht module S^μ in permutation module M^λ.

When restricting on $l = l(\lambda)$ variables. It can also be expressed as

$$s_\lambda(x_1, \ldots, x_l) = \det((h_{\lambda_i - i+j})_{l\times l}) = \frac{a_{\lambda+\delta}(x)}{a_\delta(x)},$$

The determinant in the first identity is called *Jacobi-Trudi determinant*, where h are *complete homogeneous functions*, defined by

$$h_n = \begin{cases} \sum_\mu x^\mu, & n \geq 0; \\ 0, & n < 0. \end{cases}$$

where the sum is over the set of all compositions μ of n. In the second identity $\delta := (l-1, l-2, \ldots, 0)$ and for any $\nu \in \mathbb{N}^l$, a_ν are alternating polynomials, defined by

$$a_\nu(x) = \det((x_i^{\nu_j})_{l\times l}).$$

The last quotient form of Schur functions is the original definition by Schur over 110 years

ago.

As a use of Schur function, Cauchy's identity states

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}.$$

Both sides are generating functions, but for any $m, k \in \mathbb{N}_+$ by restricting $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_k)$ and $x_i y_j \in (0, 1), \forall (i, j) \in [m] \times [k]$, it still holds:

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{i=1}^{m} \prod_{j=1}^{k} \frac{1}{1 - x_i y_j}. \tag{1.12}$$

Chapter 2

RSK with Random Input

RSK taking random input has been worked on in [BDJ99, Joh00, O'C03a, O'C03b] where two friendly distributions of the shape of output are involved. One is *Plancherel measure* of symmetric group $\mathcal{S}_n$. By picking a permutation in $\mathcal{S}_n$ uniformly at random and perform RSK correspondence on it, the push-forward law for the shape of output tableaux is the Plancherel measure.

$$\mathbb{P}(\mathrm{sh}P(n) = \lambda) = \frac{(f^\lambda)^2}{n!}.$$

As mentioned before the fact that it's a probability measure is equivalent to the identity (1.1). This choice of measure was considered in [BDJ99] and the asymptotics of $L_1^n(n) = (\mathrm{sh}P(n))_1$ was obtained (the attractor is Tracy-Widom law[TW94]).

The other friendly measure is Schur measure and corresponds to the Cauchy identity (1.12), as is constructed below.

Let the entries (ξ_{ij}) of $D \in M(\mathbb{N})_{n\times k}$ be independent random variables, then the corresponding walks $(S_i)_{1\le i\le k}$ are k independent random walks:

$$S_j(m) = \sum_{i=1}^{m} \xi_{ij}, \quad j = 1, \dots, k.$$

For $m' \ge 0$ denote

$$S_j(m, m+m') := \xi_{m,j} + \xi_{m+1,j} + \cdots + \xi_{m+m',j}$$

to be the distance the walks travel between time m and $m+m'$. Then by (1.11) $L_1^k(n) =$

$(\mathrm{sh}P(n))_1$ can be written as a path functional of of random walks:

$$L_1^k(n) = \max_{1\le i_1\le\cdots\le i_{k-1}\le n}\{S_1(1,i_1)+S_2(i_1,i_2)+\cdots+S_{k-1}(i_{k-2},i_{k-1})+S_k(i_{k-1},n)\}. \quad (2.1)$$

2.1 Dynamics

Let us calculate the dynamics of the shape. The calculation in this section follows similar cases in [O'C03a]. Since the increment ξ_{ij} in each time step of the walks are independent, it is clear that (P,Q) are markov. However, if we choose good distribution for ξ_{mi}'s, the shape of P can also be a Markov process.

Lemma 2.1. *For $(m,i)\in[n]\times[k]$ let p_m and q_i be positive numbers such that $p_mq_i<1$. Let ξ_{mi} be geometrically distributed with parameter $1-p_mq_i$, i.e.*

$$\mathbb{P}(\xi_{mi}=j)=(1-p_mq_i)(p_mq_i)^j,\quad j\in\mathbb{N}.$$

Then L^k is a (time-inhomogeneous) Markov chain on $\mathbb{N}^k$ with transition kernel from time $m-1$ to time m being

$$\Pi_L^m(\gamma,\lambda)=\left(\prod_{i=1}^k(1-p_mq_i)\right)\frac{s_\lambda(q)p_m^{|\lambda|}}{s_\gamma(q)p_m^{|\gamma|}}\mathbb{I}_{\gamma\prec\lambda},$$

where $\mathbb{I}_{\gamma\prec\lambda}$ is the indicate function of $\{\gamma\prec\lambda\}$.

Proof. Since RSK algorithm is a bijection, given two tableaux P' and Q' and knowing the dimension of the matrix, we can denote the matrix as $(\xi(P',Q')_{i,j})$. And since the ty$P(m)$ and ty$Q(m)$ corresponds to the column and row sums of the input matrix, we have a very symmetric form of the law of $(P(m),Q(m))$:

$$\begin{aligned}\mathbb{P}(P(m)=P',Q(m)=Q')&=\mathbb{P}(\xi_{li}=\xi(P',Q')_{li},1\le l\le m,1\le i\le k)\\&=\prod_{\substack{1\le l\le m\\1\le i\le k}}(1-p_lq_i)(p_lq_i)^{\xi(P',Q')_{mi}}\\&=a_m(p,q)p^{Q'}q^{P'}\mathbb{I}_{\mathrm{sh}P'=\mathrm{sh}Q'},\end{aligned}$$

where

$$a_m(p,q)=\prod_{\substack{1\le l\le m\\1\le i\le k}}(1-p_lq_i).$$

Therefore by summing up according to the shape the marginal law of each tableau can also be calculated:

$$\begin{aligned}\mathbb{P}(Q(n)=Q') &= \sum_{P':\mathrm{sh}P'=\mathrm{sh}Q'} \mathbb{P}(P(n)=P', Q(n)=Q') \\ &= \sum_{P':\mathrm{sh}P'=\mathrm{sh}Q'} a_n(p,q)p^{Q'}q^{P'} = a_n(p,q)p^{Q'}s_{\mathrm{sh}Q'}(q).\end{aligned}$$

$$\begin{aligned}\mathbb{P}(P(n)=P') &= \sum_{Q':\mathrm{sh}Q'=\mathrm{sh}P'} \mathbb{P}(Q(n)=Q', P(n)=P') &(2.2)\\ &= \sum_{Q':\mathrm{sh}Q'=\mathrm{sh}P'} a_n(q,p)q^{P'}p^{Q'} = a_n(q,p)q^{P'}s_{\mathrm{sh}P'}(p). &(2.3)\end{aligned}$$

As we already point out in (1.7), $Q(m)$ is determined by the history of shape upto time m. Hence given such a history $\lambda(1),\dots,\lambda(m)$ we can write $Q(m)$ as $Q(\lambda(1),\dots,\lambda(m))$. For $x\in\mathbb{N}^l$ and integer $m\geq l$, denote by $\mathrm{GT}^m(x)$ the set of m-level non-negative integer Gelfand-Tsetlin cones with bottom line $(x_1,\dots,x_l,0,\dots,0)$, i.e. x appended with $m-l$ 0's. Now we can compute the candidate for a "transition kernel" for L^k thanks to the friendly geometric distribution:

$$\begin{aligned}&\mathbb{P}(L^k(m+1)=\lambda|L^k(1)=\gamma(1),\dots L^k(m)=\gamma(m))\\ &=\frac{\mathbb{P}(L^k(1)=\gamma(1),\dots,L^k(m)=\gamma(m),L^k(m+1)=\lambda)}{\mathbb{P}(L^k(1)=\gamma(1),\dots,L^k(m)=\gamma(m))}\\ &=\frac{\mathbb{P}(Q(m+1)=Q(\gamma(1),\dots,\gamma(m),\lambda))}{\mathbb{P}(Q(m)=Q(\gamma(1),\dots,\gamma(m)))}\\ &=\frac{a_{m+1}(p,q)p_{1:m+1}^{Q(\gamma(1),\dots,\gamma(m),\lambda)}s_\lambda(q)}{a_m(p,q)p_{1:m}^{Q(\gamma(1),\dots,\gamma(m))}s_{\gamma(m)}(q)}\\ &=\left(\prod_{i=1}^k(1-p_{m+1}q_i)\right)\frac{s_\lambda(q)}{s_{\gamma(m)}(q)}\frac{p_{m+1}^{|\lambda|}}{p_{m+1}^{|\gamma(m)|}}\mathbb{I}_{\gamma(m)\prec\lambda}.\end{aligned}$$

And we arrive at our conclusion. □

By summing up the joint of $(P(m),Q(m))$ over fixed shape λ, the distribution of the shape of output is

$$\mathbb{P}(L^k(m)=\lambda)=\sum_{(P',Q'):\mathrm{sh}P'=\mathrm{sh}Q'=\lambda}\mathbb{P}(P(m)=P',Q(m)=Q')=a_n(p,q)s_\lambda(p)s_\lambda(q). \quad (2.4)$$

This is called *Schur measure* [Oko01]. Since it's a probability measure, by summing up over

all possible shapes we can recover the Cauchy's identity (1.12).

Chapter 3

Pitman's Theorem

Pitman's $2M - X$ theorem gives an alternative definition of 3-dimensional Bessel process.

3.1 3-Dimensional Bessel Process

Definition 3.1. A *d-dimensional Bessel process* Bes^d starting at x is defined as the radial part of d-dimensional Brownian motion starting at y with $|y| = x$, i.e. $(X_t, t \geq 0) \overset{\mathbf{d}}{=} (\sqrt{B_1(t)^2 + \cdots + B_d(t)^2}, t \geq 0)$, where $B_1, \ldots, B_d$ are d independent Brownian motions with $(B_1, \ldots, B_d)(0) = y$.

Bessel processes get their names due to the connection to modified Bessel functions with their transition kernels. The 3-dimensional Bessel process is of interest due to its various interpretations. One construction is by conditioning a one-dimensional Brownian motion never to hit 0 after time 0 in Doob's sense. Another, due to Pitman [Pit75], is one of the focuses of this dissertation.

Theorem 3.2 (Pitman's Theorem)**.** *Let B be a 1-dimensional Brownian motion and M be its past maximum: $M_t = \sup_{s \leq t} B_t$. Then path transform $(2M_t - B_t, t \geq 0)$ is a 3-dimensional Bessel process and satisfies SDE*

$$dX_t = dW_t + \frac{1}{X_t} dt.$$

The mere fact that $2M - X$ is Markov is already a miracle.

3.2 Brownian Motion with a Constant Drift

Generally for any one-dimensional diffusion process X, define a path transform $\mathfrak{P}$ by

$$\mathfrak{P}X(t) := 2M(t) - X(t) = 2\sup_{s\le t} X(s) - X(t);$$

This transform reflects the path on its previous maximum which is generalised as Pitman transform in [BBO05]. It is a problem of interest whether $\mathfrak{P}X$ is still Markov and what its transition kernel or generator can be. In [RP81] Rogers and Pitman extend Pitman's theorem to X being a one-dimensional Brownian motion with a constant drift c:

$$X_t = B^c = x + B_t + ct.$$

They achieve this by using a criterion that a function ϕ on a given Markov process X is still Markov, which itself is important. Suppose the state space of X is $(S, \mathcal{S})$ and function $\phi : (S, \mathcal{S}) \to (S', \mathcal{S}')$. Define kernel Φ by $\Phi f = f \circ \phi$. The criterion is

Theorem 3.3 ([RP81])**.** *Let X be a Markov process on $(S, \mathcal{S})$ with kernel $(P_t)_{t\ge 0}$. Let $\Lambda : S \to \mathcal{M}S$ be a kernel and $(Q_t)_{t\ge 0}$ be a family of kernels on S'. If the following conditions are satisfied,*

1. *$\Lambda\Phi = I$, the identity kernel on $C_b(S')$,*
2. *$\Lambda P_t = Q_t\Lambda$.*

then

$$\mathbb{P}(X_0 \in A | \phi(X_0) = y) = \Lambda(y, A)$$

implies

$$\mathbb{P}(X_t \in A | \phi(X_t) = y; \phi(X_s), 0 \le s \le t) = \Lambda(y, A). \tag{3.1}$$

Note (3.1) means $\phi(X)$ is a Markov process with kernel (Q_t). The second condition is the so called intertwining relationship between (P_t) and (Q_t) and we call Λ the *intertwining kernel*. This formula is used extensively in literature.

In the context of determining wether $\mathfrak{P}(B^c)$ is Markov and finding its dynamics, [RP81] examines $X = (M^c - B^c, M^c)$ and $\phi(x) = x_1 + x_2$, works out its intertwining kernel Λ^c:

$$\Lambda^c(y, (u, v)) \propto e^{c(v-u)},$$

and applies Theorem 3.3.

Theorem 3.4 (Pitman's Theorem^c)**.** *The process $\mathfrak{P}(B^c)$ is Markov and has the same distribution as the radial part of a 3-dimensional Brownian motion with a drift with magnitude $|c|$, denoted as $Bes^{3,c}$. It satisfies the SDE*

$$dX_t = dW_t + c \coth cX_t dt.$$

Moreover, like the Brownian motion case, $\text{Bes}^{3,c}$ can be constructed by conditioning on B^c starting at x never hitting 0.

Chapter 4

Scaling from RSK to Pitman

In this chapter we attempt to prove Pitman's theorem for more generalised case: Brownian motion with a variable drift in a similar way as the proof of Theorem 3.4. Thanks to Theorem 3.3 we would be halfway to success if we can guess the intertwining kernel. We are going to achieve it via scaling RSK with random input. We first test this method by recovering Λ^c and then move on to Brownian motion with variable drift in the same way. The former is done in a non-rigorous way since we are recovering a known result.

4.1 The Plan

Now set $k = 2$, then $L^k = \mathrm{sh}P$ has two entries. By (2.1)

$$L_1^2(m) = S_2(m) + \max_{1\le l\le m}\{S_1(l) - S_2(l-1)\};$$
$$L_2^2(m) = S_1(m) + \min_{1\le l\le m}\{S_2(l-1) - S_1(l)\};$$

Therefore by taking

$$A = S_1 - S_2 \text{ and } M(m) = \max_{l\le m} X(l);$$

we have

$$\begin{aligned}(L_1^2 - L_2^2)(m) &= 2\max_{1\le l\le m}\{S_1(l) - S_2(l-1)\} - (S_1(m) - S_2(m)) \\ &= (2M - A)(m) + \epsilon(m) \approx (2M - A)(m), \qquad (4.1)\end{aligned}$$

where

$$\epsilon(m) = 2 \max_{1\le l\le m}\{S_1(l) - S_2(l-1)\} - 2 \max_{1\le l\le m}\{S_1(l) - S_2(l)\}.$$

When m is large enough the terms $\epsilon(m)$ is insignificant compared to other accumulative terms like $A(m)$ and $M(m)$, thus should tend to 0 in suitable sense after scaling. Therefore we are going to neglect it at some stage.

This seems to be a discrete version of transform $\mathfrak{P}$. Therefore by Donsker's invariance principle, we can scale the discrete object to obtain Pitman's theorem for various diffusion processes. For any discrete-time process $(X(n), n \ge 0)$, its *diffusion scaling limit* is a continuous-time process $(\chi(t), t \ge 0)$ defined by

$$(\chi_t)_{t\ge 0} = \lim_{N\to\infty} \left(\frac{X(Nt)}{\sqrt{N}}\right)_{t\ge 0} \text{ in distribution.} \tag{4.2}$$

Now denote scaling limit of A and M by α and μ respectively. Our first step is to identify them after picking $(p_i)_{i=1,\dots,k}$ $(q_j)_{j=1,2}$ carefully. Then by (4.1), $\mathfrak{P}\alpha$ is the scaling limit of $L_1^2 - L_2^2$. We can expect from the fact L is a Markov chain (Lemma 2.1) that α should also be Markov. To emulate the method used in [RP81] to reveal the dynamics of $\mathfrak{P}B^c$ we need to guess the intertwining kernel Λ defined by

$$\Lambda_t^c(x,(u,v))dudv := \mathbb{P}((M^c - B^c, M^c)(t) \in (du,dv) | \mathfrak{P}B^c(t) = x, \mathfrak{P}B^c(s) : 0 \le s < t).$$

Therefore the second step is to infer the same kernel in discrete setting

$$\tilde{\Lambda}_m(x,(u,v)) := \mathbb{P}((M-A,M)(m) = (u,v) | (L_1^2 - L_2^2)(m) = x, (L_1^2 - L_2^2)(i) : 0 \le i < m).$$

And once this is done we can verify the intertwining relationship, drop RSK setting and prove it rigorously.

4.2 Choosing Parameters

In our model, parameters of the distribution of the entries of the input matrix have full control of what we can expect of α, therefore we want to pick p_m's and q_i's to make α the process we want. Let us start with the Brownian motion with a constant drift, the case already studied and proved in [RP81] and mentioned in the previous chapter.

Lemma 4.1. *Given a $c > 0$, for every N in the definition of scaling limit* (4.2)*, let*

$$\begin{aligned} p_i &= p := (4 - 2\sqrt{3})c, \quad \forall i; \\ q_1 &= \frac{1}{2c} + \frac{1}{\sqrt{N}}, \quad q_2 = q := \frac{1}{2c}; \end{aligned}$$

then

$$\alpha \overset{\mathbf{d}}{=} B^c.$$

Proof. The proof is quite straightforward. We specify the mean and variance of $\xi_{i1} - \xi_{i2}$ and thus find the scaling limit for the "difference" random walk $A(m) = \sum_{i=1}^{m}(\xi_{i1} - \xi_{i2})$. The mean and variance for a geometric distribution with parameter $1 - pq_i$ are $\frac{pq_i}{1-pq_i}$ and $\frac{pq_i}{(1-pq_i)^2}$. Therefore

$$\begin{aligned} \mathbb{E}(\xi_{i1} - \xi_{i2}) &= \frac{pq_1}{1-pq_1} - \frac{pq_2}{1-pq_2} \\ &= \frac{p(q_1 - q_2)}{(1-pq_1)(1-pq_2)} = \frac{1}{\sqrt{N}} \frac{p}{(1 - p(q + \frac{1}{\sqrt{N}}))(1-pq)} \\ &= \frac{1}{\sqrt{N}} \frac{p}{(1-pq)^2} + \frac{1}{N} \frac{p}{(1-pq)^2(1 - p(q + \frac{1}{\sqrt{N}}))} \\ &=: \frac{1}{\sqrt{N}} \frac{p}{(1-pq)^2} + \frac{1}{N}\delta_1(N) = \frac{1}{\sqrt{N}}c + \frac{1}{N}\delta_1(N). \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathbb{V}\mathrm{ar}(\xi_{i1} - \xi_{i2}) &= \mathbb{V}\mathrm{ar}\xi_{i1} + \mathbb{V}\mathrm{ar}\xi_{i2} = \frac{p(q + \frac{1}{\sqrt{N}})}{(1 - p(q + \frac{1}{\sqrt{N}}))^2} + \frac{pq}{(1-pq)^2} \\ &= \frac{2pq}{(1-pq)^2} + \frac{p(q + \frac{1}{\sqrt{N}})}{(1 - p(q + \frac{1}{\sqrt{N}}))^2} - \frac{pq}{(1-pq)^2} \\ &= \frac{2pq}{(1-pq)^2} + \frac{1}{\sqrt{N}} \frac{p}{(1-pq)^2} \frac{1 - q(q + \frac{1}{\sqrt{N}})p^2}{(1 - p(q + \frac{1}{\sqrt{N}}))^2} \\ &=: \frac{2pq}{(1-pq)^2} + \frac{1}{\sqrt{N}}\delta_2(N) = 1 + \frac{1}{\sqrt{N}}\delta_2(N). \end{aligned} \tag{4.4}$$

We'll show that in the scaling limit, the terms $\frac{1}{N}\delta_1$ and $\frac{1}{\sqrt{N}}\delta_2$ can be ignored, which is quite intuitive. Denote $\hat{A}$ as the normalisation of A:

$$\hat{A}(m) = \frac{A(m) - m\mathbb{E}(\xi_{i1} - \xi_{i2})}{\sqrt{\mathbb{V}\mathrm{ar}(\xi_{i1} - \xi_{i2})}},$$

then by Donsker's theorem, $\hat{A}$ scales into a one-dimensional Brownian motion, that is

$$\left(\frac{A(Nt) - Nt\mathbb{E}(\xi_{i1} - \xi_{i2})}{\sqrt{N}\sqrt{\mathbb{V}\mathrm{ar}(\xi_{i1} - \xi_{i2})}}, t \geq 0 \right) \to B^0.$$

On the right hand side,

$$\begin{aligned}
\frac{A(Nt) - Nt\mathbb{E}(\xi_{i1} - \xi_{i2})}{\sqrt{N}\sqrt{\mathbb{V}\mathrm{ar}(\xi_{i1} - \xi_{i2})}} &= \frac{A(Nt) - Nt(\frac{1}{\sqrt{N}}c + \frac{1}{N}\delta_1(N))}{\sqrt{N}\sqrt{1 + \frac{1}{\sqrt{N}}\delta_2(N)}} \\
&= \frac{1}{1 + \frac{1}{\sqrt{N}}\delta_2(N)} \left(\frac{A(Nt)}{\sqrt{N}} - ct \right) - \frac{1}{\sqrt{N}} \frac{t\delta_1(N)}{1 + \frac{1}{\sqrt{N}}\delta_2(N)}.
\end{aligned}$$

So it suffices to show that δ_1 and δ_2 are both bounded, which is trivial. For N large enough,

$$\begin{aligned}
0 \leq \delta_1(N) &= \frac{p}{(1-pq)^2} \frac{1}{1 - p(q + \frac{1}{\sqrt{N}})} \leq \frac{p}{(1-pq)^2} \frac{1}{1-2pq}. \\
0 \leq \delta_2(N) &= \frac{p}{(1-pq)^2} \frac{1 - q(q + \frac{1}{\sqrt{N}})p^2}{(1 - p(q + \frac{1}{\sqrt{N}}))^2} \leq \frac{p}{(1-pq)^2} \frac{1 - q^2p^2}{(1-2pq)^2}.
\end{aligned}$$

□

4.3 The Intertwining Kernel

In order to find the intertwining kernel Λ^c, we need to clarify the links between the information given by the tableau obtained from RSK algorithm and the various quantities of interest. We suppress the time arguments of each process.

Lemma 4.2. *Rewrite the tableau P as $P = (P_1, P_2, P_3)$, where $P_1 := L_1^1$ and $P_2 := L_1^2 - L_1^1$ are the number of 1's and 2's in the first row of P respectively and $P_3 := L_2^2$ is the number of $2's$ (i.e. the length) in the second row. Then*

$$\begin{aligned}
S_1 = P_1, \qquad & S_2 = P_2 + P_3, \qquad & A = P_1 - P_2 - P_3, \\
M - A = P_2 + \frac{\epsilon}{2}, \qquad & M = P_1 - P_3 + \frac{\epsilon}{2}, \qquad & 2M - A = P_1 + P_2 - P_3 + \epsilon.
\end{aligned}$$

Proof. The random walk S_1 is the 1st column sum of the input matrix which corresponds to the number of 1's in the P-tableau, hence

$$S_1 = P_1. \tag{4.5}$$

Because the sum of two random walks is the same as the sum of all entries of D which coincides with the sum of P_1, P_2 and P_3 as the number of boxes. Hence by (4.5),

$$S_2 = P_2 + P_3. \tag{4.6}$$

By the definition of A and (4.6),

$$A = S_1 - S_2 = P_1 - P_2 - P_3. \tag{4.7}$$

By (4.1) and since P_1 and P_2 represent all the boxes in row 1 and P_3 for row 2,

$$2M - A = L_1^2 - L_2^2 + \epsilon = P_1 + P_2 - P_3 + \epsilon. \tag{4.8}$$

Combining (4.7) with (4.8) we obtain the remaining two equalities. □

Now we are ready to calculate the discrete intertwining kernel. First using the formula (2.2) of distribution of P, we have for valid (P_2', x, y),

$$\begin{aligned}\mathbb{P}(P_2(m) = P_2', L_1^2(m) + L_2^2(m) = x, L_1^2(m) - L_2^2(m) = y) \\ = a_m(p,q) q_1^{P_1(m)} q_2^{P_2(m)+P_3(m)} s_{(\frac{x+y}{2}, \frac{x-y}{2})}(p).\end{aligned}$$

Since

$$\begin{aligned}P_1 + P_2 = L_1^2 = \frac{L_1^2 + L_2^2}{2} + \frac{L_1^2 - L_2^2}{2}; \\ P_3 = L_2^2 = \frac{L_1^2 + L_2^2}{2} - \frac{L_1^2 - L_2^2}{2},\end{aligned}$$

we can express P_1 and $P_2 + P_3$ in terms of $P_2, L_1^2 + L_2^2$ and $L_1^2 - L_2^2$.

$$\begin{aligned}P_1 = \frac{1}{2}((L_1^2 + L_2^2) + (L_1^2 - L_2^2)) - P_2; \\ P_2 + P_3 = P_2 + \frac{1}{2}((L_1^2 + L_2^2) - (L_1^2 - L_2^2)).\end{aligned}$$

Also to reduce mess we define polynomial $\tilde{s}_{x,y}$ by

$$\tilde{s}_{x,y}(p) = s_{(\frac{x+y}{2}, \frac{x-y}{2})}(p).$$

Therefore the joint of $(P_2, L_1^2 + L_2^2, L_1^2 - L_2^2)$ can rewritten as

$$\mathbb{P}(P_2(m) = P_2', L_1^2(m) + L_2^2(m) = x, L_1^2(m) - L_2^2(m) = y) = a_m(p,q) q_1^{\frac{x+y}{2} - P_2'} q_2^{\frac{x-y}{2} + P_2'} \tilde{s}_{x,y}(p).$$

Dnote by D_y (or $D_{P_2',y}$) the sets of (P_2', x) (or x) such that (P_2', x, y) as $(P_2, L_1^2 + L_2^2, L_1^2 - L_2^2)$ can form a valid P-tableau. Dividing the previous formula by summation over $\{L_1^2 - L_2^2 = y\}$ we can obtain the conditional law of $(P_2, L_1^2 + L_2^2)$ given $L_1^2 - L_2^2$:

$$\begin{aligned}\mathbb{P}(P_2 = P_2', L_1^2 + L_2^2 = x | L_1^2 - L_2^2 = y) &= \frac{a_m(p,q) q_1^{\frac{x+y}{2} - P_2'} q_2^{\frac{x-y}{2} + P_2'} \tilde{s}_{x,y}(p)}{\sum_{(P_2', x) \in D_y} a_m(p,q) q_1^{\frac{x+y}{2} - P_2'} q_2^{\frac{x-y}{2} + P_2'} \tilde{s}_{x,y}(p)} \\ &= \frac{q_1^{\frac{x+y}{2} - P_2'} q_2^{\frac{x-y}{2} + P_2'} \tilde{s}_{x,y}(p)}{\sum_{(P_2', x) \in D_y} q_1^{\frac{x+y}{2} - P_2'} q_2^{\frac{x-y}{2} + P_2'} \tilde{s}_{x,y}(p)} =: Z_y q_1^{\frac{x+y}{2} - P_2'} q_2^{\frac{x-y}{2} + P_2'} \tilde{s}_{x,y}(p),\end{aligned}$$

By Lemma 4.2 (note we are not doing rigorous calculation in (4.9) by neglecting ϵ),

$$\begin{aligned}\tilde{\Lambda}_m(y, (P_2', z)) &= \mathbb{P}(M - A = P_2', M = z | 2M - A = y) \\ &= \mathbb{I}_{P_2' + z = y} \mathbb{P}\left(P_2 + \frac{\epsilon}{2} = P_2' | L_1^2 - L_2^2 + \epsilon = y\right) \\ &= \mathbb{I}_{P_2' + z = y} \mathbb{P}(P_2 = P_2' | L_1^2 - L_2^2 = y) \qquad (4.9) \\ &= \mathbb{I}_{P_2' + z = y} \sum_{x \in D_{P_2', y}} \mathbb{P}(P_2 = P_2', L_1^2 + L_2^2 = x | L_1^2 - L_2^2 = y) \\ &= \mathbb{I}_{P_2' + z = y} Z_y \sum_{x \in D_{P_2', y}} q_1^{\frac{x+y}{2} - P_2'} q_2^{\frac{x-y}{2} + P_2'} \tilde{s}_{x,y}(p). \qquad (4.10)\end{aligned}$$

By Lemma 4.2

$$P_2 = \frac{(P_1 + P_2 - P_3) - (P_1 - P_2 - P_3)}{2} = \frac{(L_1^2 - L_2^2) - A}{2},$$

thus the difference of random walk A' corresponding to the P-tableau formed by (P_2', x, y) as $(P_2, L_1^2 + L_2^2, L_1^2 - L_2^2)$ satisfies

$$P_2' = \frac{y - A'}{2}. \qquad (4.11)$$

We can also take a close look at the valid domain for x under the summation. It means (P_2', x, y) as $(P_2, L_1^2 + L_2^2, L_1^2 - L_2^2)$ can form a valid tableau

$$(P_1', P_2', P_3') = (\frac{x+y}{2} - P_2', P_2', \frac{x-y}{2}), \qquad (4.12)$$

i.e.

$$P_2' \geq 0; \quad P_1' \geq P_3' \geq 0,$$

which by (4.12) is equivalent to

$$x \geq y \geq P_2' \geq 0.$$

Since P_2' and y are fixed in the summation (4.10), the valid domain for x is $[y, \infty) \cap \mathbb{N}$. Hence by (4.10) and (4.11)

$$\begin{aligned}
\tilde{\Lambda}_{Nt}(y, (P_2', z)) &= \mathbb{I}_{P_2'+z=y} Z_y \sum_{x \geq y} q_1^{\frac{x+y}{2} - \frac{y-A'}{2}} q_2^{\frac{x-y}{2} + \frac{y-A'}{2}} \tilde{s}_{x,y}(p) \\
&= \mathbb{I}_{P_2'+z=y} \tilde{Z}_y q_1^{\frac{A'}{2}} q_2^{-\frac{A'}{2}} = \tilde{Z}_y \left(\frac{q + \frac{1}{\sqrt{N}}}{q} \right)^{\frac{A'}{2}} \\
&= \mathbb{I}_{P_2'+z=y} \tilde{Z}_y \left(1 + \frac{1}{q\sqrt{N}} \right)^{q\sqrt{N} \cdot \frac{1}{2q} \cdot \frac{A'}{\sqrt{N}}} \propto \mathbb{I}_{P_2'+z=y} \left(1 + \frac{1}{q\sqrt{N}} \right)^{q\sqrt{N} c \frac{z-P_2'}{\sqrt{N}}}.
\end{aligned}$$

What remains to be done is a scaling. Although we can already see the natural exponential hidden in the limit, let us finish it a bit more clearly (but not rigorously),

$$\begin{aligned}
\Lambda_t^c(y, (u, v)) &= \mathbb{P}(\mu_t - \alpha_t \in du, \mu_t \in dv | 2\mu_t - \alpha_2 = y)/(dudv) \\
&= \lim_{\epsilon \to 0} \frac{1}{4\epsilon^2} \mathbb{P}(\mu_t - \alpha_t \in (u-\epsilon, u+\epsilon], \mu_t \in (v-\epsilon, v+\epsilon] | 2\mu_t - \alpha_2 = y) \\
&= \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{4\epsilon^2} \mathbb{P}\left(\frac{(M-A, M)(Nt)}{\sqrt{N}} \in S_\epsilon(u, v) \Big| \frac{(2M-A)(Nt)}{\sqrt{N}} = y \right) \\
&= \lim_{N \to \infty} \lim_{\epsilon_N \to 0} \frac{1}{4\epsilon_N^2} \mathbb{P}\left(\frac{(M-A, M)(Nt)}{\sqrt{N}} \in S_{\epsilon_N}(u, v) \Big| \frac{(2M-A)(Nt)}{\sqrt{N}} = y \right),
\end{aligned}$$

where $S_{\epsilon_N}(u, v) = (u - \epsilon_N, u + \epsilon_N] \times (v - \epsilon_N, v + \epsilon_N]$ is a square centred at u, v with side length $2\epsilon_N$. For each N, by taking $\epsilon_N = \frac{1}{2\sqrt{N}}$, there can be only one integer point in $S_{\sqrt{N}\epsilon_N}(\sqrt{N}u, \sqrt{N}v)$. And since the distance between this point and the centre $(\sqrt{N}u, \sqrt{N}v)$ is no larger than $\sqrt{2}$ for all N, for similicity we can identify them with each other. Hence the kernel is now

$$\begin{aligned}
\Lambda_t(y, (u, v)) &= \lim_{N \to \infty} N\mathbb{P}\left((M-A, M)(Nt) \in S_{\sqrt{N}\epsilon_N}(\sqrt{N}u, \sqrt{N}v) \Big| (2M-A)(Nt) = \sqrt{N}y \right) \\
&= \lim_{N \to \infty} N\mathbb{P}\left((M-A, M)(Nt) = (\sqrt{N}u, \sqrt{N}v) \Big| (2M-A)(Nt) = \sqrt{N}y \right) \\
&= \lim_{N \to \infty} N\Lambda_{Nt}^d\left(\sqrt{N}y, (\sqrt{N}u, \sqrt{N}v) \right) = \lim_{N \to \infty} N\tilde{Z}_{N,y} \left(1 + \frac{1}{q\sqrt{N}} \right)^{q\sqrt{N}c(u-v)} \\
&= (\lim_{N \to \infty} N\tilde{Z}_{N,y}) e^{c(u-v)} \propto e^{c(u-v)}.
\end{aligned}$$

This is exactly what was obtained in proof of Theorem 1 in [RP81].

4.4 Any Luck in Generalisation?

In the previous sections, by making p_m's to be constant, we are able to recover the Pitman's theorem for a Brownian motion with a constant drift. So one may think, by letting parameters vary, we may be able to prove Pitman theorem for more general diffusion processes. Unfortunately this is an illusion.

For any given function f, let us evaluate the parameters with a general setting

$$p_m = f\left(\frac{m}{N}\right), \quad q_1 = q + \frac{1}{N}, q_2 = q;$$

and see what happens to the scaling limit. As in the proof of Lemma 4.1, the mean and variance of the difference $\xi_{i2} - \xi_{i2}$ is (just replace p with p_i in (4.3) and (4.4))

$$\begin{aligned}
\mathbb{E}(\xi_{i1} - \xi_{i2}) &= \frac{1}{\sqrt{N}} \frac{p_i}{(1-p_i q)^2} + \frac{1}{N} \frac{p_i}{(1-p_i q)^2 (1 - p_i(q + \frac{1}{\sqrt{N}}))}, \\
\mathbb{V}\mathrm{ar}(\xi_{i1} - \xi_{i2}) &= \frac{2p_i q}{(1-p_i q)^2} + \frac{1}{\sqrt{N}} \frac{p_i}{(1-p_i q)^2} \frac{1 - q(q + \frac{1}{\sqrt{N}}) p_i^2}{(1 - p_i(q + \frac{1}{\sqrt{N}}))^2}.
\end{aligned}$$

We don't need to work rigorously, as all we want is a guess of the kernel Λ, verify the intertwining relationship and use Theorem 3.3 to achieve the result. Therefore we discard the terms with lower order:

$$\begin{aligned}
\mathbb{E}(\xi_{i1} - \xi_{i2}) &\sim \frac{1}{\sqrt{N}} \frac{f(\frac{i}{N})}{(1 - f(\frac{i}{N}) q)^2}, \\
\mathbb{V}\mathrm{ar}(\xi_{i1} - \xi_{i2}) &\sim \frac{2 f(\frac{i}{N}) q}{(1 - f(\frac{i}{N}) q)^2}.
\end{aligned}$$

Hence by Donsker's theorem, the scaling limit α should satisfy the SDE:

$$d\alpha_t = \sqrt{\frac{2qf(t)}{(1-qf(t))^2}} dW_t + \frac{f(t)}{(1-qf(t))^2} dt.$$

where dW_t is Brownian motion. But what we want is a Brownian motion with a drift, so let us perform a deterministic time change, define an increasing function $T : [0, \infty) \to [0, \infty)$ by

the quadratic variation of α

$$T(t) = [\alpha]_t = \int_0^t 2q \frac{f(s)}{(1 - qf(s))^2} ds,$$

and time-changed process $\tilde{\alpha}$ by

$$\tilde{\alpha}_{T(t)} = \alpha_t.$$

Then since

$$\begin{aligned} [\tilde{\alpha}]_t &= [\alpha]_{T^{-1}(t)} = [\alpha]_s \big|_{[\alpha]_s = t} = t; \\ \mathbb{E}\tilde{\alpha}_t &= \mathbb{E}\alpha_{T^{-1}(t)} = \int_0^{T^{-1}(t)} \frac{f(s)}{(1 - qf(s))^2} ds = \int_0^t \frac{1}{2q} du, \end{aligned}$$

$\tilde{\alpha}$ should be a Brownian motion with a drift $\frac{1}{2q}$. Therefore whatever f we choose, we end up with the constant drift case. Thus even if we are able to guess the intertwining kernel, it is not likely to lead to Pitman's theorem for more general processes using this RSK setting, unfortunately.

Chapter 5

Extensions of Pitman and RSK

Till now we have only focused on 2 dimensional discrete and continuous Pitman's Theorem, but there are more, so is true for RSK correspondence, which we only considered multidimensional discrete case. In this chapter a survey of generalisation of Pitman's theorem and RSK correspondence to multi-dimension, geometric lifting and continuum models is presented.

5.1 Multi-dimension

In the preceeding two chapters we fix $k = 2$ and connect Pitman's theorem with a path transform of random walks generated by RSK correspondence

$$\begin{aligned} L_1^2(m) &= \max_{0\le l\le m}\{S_1(l) + S_2(m) - S_2(l-1)\};\\ L_2^2(m) &= \min_{0\le l\le m}\{S_2(l-1) + S_1(m) - S_1(l)\}. \end{aligned}$$

If we consider RS correspondence on words (Example 1.2), then since each row of input matrix has only 1 non-zero entry, the above formula can be rewritten as:

$$\begin{aligned} L_1^2(m) &= \max_{0\le l\le m}\{S_1(l) + S_2(m) - S_2(l)\};\\ L_2^2(m) &= \min_{0\le l\le m}\{S_2(l) + S_1(m) - S_1(l)\}. \end{aligned} \tag{5.1}$$

This leads to the definition of operators $\triangle$ and $\triangledown$, constructed in [O'C03b]:

$$\begin{aligned} (x \triangle y)(m) &:= \min_{0\le l\le m}\{x(l) + y(m) - y(l)\};\\ (x \triangledown y)(m) &:= \max_{0\le l\le m}\{x(l) + y(m) - y(l)\}; \end{aligned}$$

Let the order of transform be from left to right in a sequence of operations, e.g.

$$x_1 \triangle x_2 \triangledown x_3 \triangledown x_4 = ((x_1 \triangle x_2) \triangledown x_3) \triangledown x_4.$$

Then for general k a path tranform G^k for discrete-time process can be defined:

$$\begin{aligned} G^1(x) &= x; \\ G^2(x,y) &= (x \triangle y, y \triangledown x); \\ G^k(x_1,\ldots,x_k) &= (x_1 \triangle \ldots \triangle x_k, G^{k-1}(x_2 \triangledown x_1, x_3 \triangledown (x_1 \triangle x_2), \ldots, x_k \triangledown (x_1 \triangle \ldots \triangle x_{k-1}))). \end{aligned}$$

For convenience we write $\cdot(\cdot)^r := (\cdot(\cdot))^r$, for example we write $G^k(S)^r$ instead of $(G^k(S))^r$. Immediately by (5.1) we have given a word $w \in [2]^n$,

$$\mathrm{sh}P(w) = G^2(S^r)^r,$$

For general k, there's a connection between dual RSK correspondence and the path transform:

Theorem 5.1 (Theorem 3.1 in [O'C03b])**.** *Given a word $w \in [k]^n$ and its associated walk $S(w)$,*

$$shP'(w) = G^k(S(w))^r. \tag{5.2}$$

This is proved by constructing "a series of queues in series" out of the walk and connecting the structure of $P'(w)$ and $G^k(S)$ by interpreting both in the dynamics of the queues.

Then it can be readily shown:

Corollary 5.2. *Given a word $w \in [k]^n$ and its associated walk $S(w)$,*

$$shP(w) = G^k(S(w)^r)^r.$$

Proof. Let us define a transform on words $^* : [k]^n \to [k]^n$ by

$$w^* = (k+1-w_n, k+1-w_{n-1}, \ldots, k+1-w_1).$$

We explain this tranformation threefold: as for the word w it reverts both the word and the alphabet; as for w's associated input matrix $D(w)$, it rotates the matrix for 180 degrees around its "centre" (also two reversion); as for the walk $S(w)$, it reverts the time as well as the index of random walks. It turns out the input matrix interpretation is the most

proof-friendly, as we can decompose the transform into two:

$$w(D)^* = w(hv(D)) \tag{5.3}$$

where h and v reverts the rows and columns of the matrix. Then immediate we have for any matrix C,

$$w(h(C)) = w(C)^r; \tag{5.4}$$
$$S(v(C)) = S(C)^r. \tag{5.5}$$

On the other hand, a duality theorem (see e.g. Section A.1 of [Ful97]) reveals that

$$\mathrm{sh}P(w) = \mathrm{sh}P(w^*). \tag{5.6}$$

That is, this transform conserves the shape of P-tableaux! This is quite amazing. Therefore

$$\begin{aligned}\mathrm{sh}P(w) &\overset{(5.6)}{=} \mathrm{sh}P(w^*) \overset{(5.3)}{=} \mathrm{sh}P(w(h(v(D)))) \overset{(5.4)}{=} \mathrm{sh}P(w(v(D))^r) \\ &\overset{(1.3)}{=} \mathrm{sh}P'(w(v(D))) \overset{(5.2)}{=} G^k(S(v(D)))^r \overset{(5.5)}{=} G^k(S^r)^r.\end{aligned}$$

□

As a side note, the conclusion of the preceding Corollary is a bit clumsy in that two reverting are involved on the right hand side. To make it look better, just define a similar family of transforms H^k by

$$\begin{aligned}H^1(x) &= x; \\ H^2(x,y) &= (x \triangledown y, y \triangle x); \\ H^k(x_1,\ldots,x_k) &= (H^{k-1}(x_1 \triangledown (x_k \triangle \ldots \triangle x_2), x_2 \triangledown (x_k \triangle \ldots \triangle x_3), \ldots, x_{k-1} \triangledown x_k), x_k \triangle \ldots \triangle x_1).\end{aligned}$$

Then it can be easily verified that $H^k(x) = G^k(x^r)^r$ and hence $\mathrm{sh}P(w) = H^k(S(w))$.

The path transform is connected to conditioned random walks when RS correspondence takes random input. Denote $W := \{x \in \mathbb{R}^k : x_1 \le \cdots \le x_k\}$ as the k-dimensional Wyel chamber and $\Omega := \{x \in \mathbb{R}^k : x^r \in W\}$. Also denote by $(p_1,\ldots,p_k)$ the distribution of increment of the random walk, that is, $\mathbb{P}(S(n) - S(n-1) = e_i) = p_i$. [O'C03a] and [O'C03b] show similar results concerning the law of $\mathrm{sh}P(S)$ and $\mathrm{sh}P'(S)$.

Theorem 5.3 ([O'C03a]). *If $p_1 > p_2 > \cdots > p_k > 0$, then shP$(S)$ has the same law as S conditioned on never exiting Ω.*

Theorem 5.4 ([O'C03b]). *If $p_1 > p_2 > \cdots > p_k > 0$, then $shP'(S)$ has the same law as S conditioned on never exiting Ω.*

They are the same result since $P'(S) = P'(S(w)) = P(S(w^r))$ and that $S(w^r) \stackrel{\mathbf{d}}{=} S(w)$ due that fact that reverting the word amounts to reverting time. As a corollary, we reach the conclusion of discrete k-dimensional Pitman's theorem:

Corollary 5.5 ([O'C03b]). *For a k-dimensional random walk S with $0 < p_1 < p_2 < \cdots < p_k$, the path transform $G^k(S)$ has the same law as S conditioned on never exiting W.*

Proof. We prove it in a different way from [O'C03b]. Since by Theorem 5.3.

$$\mathrm{sh}P(S^r) \stackrel{\mathbf{d}}{=} S^r \text{ conditioned on never exiting } \Omega,$$

by reverting it,

$$\mathrm{sh}P(S^r)^r \stackrel{\mathbf{d}}{=} S \text{ conditioned on never exiting } W,$$

the conclusion is evident by Corollary 5.2. □

The conditionings above can be constructed using discrete Doob h-transform, see e.g. [O'C03a].

In the continuous setting for $f, g \in D_0(\mathbb{R}_+)$ where $D_0 = \{f \text{ a cadlag } : f(0) = 0\}$, we define operators $\otimes$ and $\odot$ by

$$(f \otimes g)(t) = \inf_{0 \le s \le t} \{f(s) + g(t) - g(s)\};$$
$$(f \odot g)(t) = \sup_{0 \le s \le t} \{f(s) + g(t) - g(s)\}.$$

Again let the default order of the operations be from left to right. Define a family of path transforms $\Gamma^k : D_0(\mathbb{R}_+)^k \to D_0(\mathbb{R}_+)^k$ recursively by

$$\Gamma^2(f_1, f_2) = (f_1 \otimes f_2, f_2 \odot f_1);$$
$$\Gamma^k(f_1, \ldots, f_k) = (f_1 \otimes f_2 \otimes \cdots \otimes f_k, \Gamma^{k-1}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \ldots, f_k \odot (f_1 \otimes \cdots \otimes f_{k-1}))).$$

Let $\tilde{B}$ be a k-dimensional Brownian motion starting in W killed upon reaching $\partial W = W \backslash \mathring{W} = W \backslash \{x \in \mathbb{R}^k : x_1 < \cdots < x_k\}$ after time 0. Let $\hat{B}$ be constructed by Doob-h transforming $\tilde{B}$ with positive (on $\mathring{W}$) harmonic function h defined as $h(x) = \prod_{i<j}(x_j - x_i)$. $\hat{B}$ is the so-called Dyson's k-dimensional non-colliding Brownian motion.

Such a construction can be found in, for example [O'C04, War07]. This process satisfies a system of SDE's:

$$d\hat{B}_t^i = dW_t^i + \sum_{j \neq i} \frac{1}{\hat{B}_t^i - \hat{B}_t^j} dt.$$

In [OY02] a sequence of stationary M/M/1 queues are considered. Let $N_1, \ldots, N_k$ be independent poisson processes with rate $\mu_1 < \cdots < \mu_k$ respectively, they obtain a conclusion similar to Corollary 5.5.

Theorem 5.6 ([OY02])**.** *The path transform $\Gamma^k(N)$ has the same law as N conditioned on never exiting W.*

To prove this, they first analyse a sequence of stationary M/M/1 queues in a similar way as the first series of queues in [O'C03b]. The difference is, the queues are are processe on $\mathbb{R}$ rather than the non-negatives. This allows them them derive a symmetry result of t processes and hence the theorem. They then use a convergence argument to show the same result for "homogeneous" queues $(N^1, \ldots, N^1)$. After that, by applying Donsker's theorem, they arrive at the conclusion for Brownian motion.

Theorem 5.7 ([OY02])**.** *For a k-dimensional Brownian motion B,*

$$\Gamma^k(B) \overset{\mathbf{d}}{=} \hat{B}.$$

Actually, it is well-known that $\hat{B}$ has the law of eigenvalues of a Hermition Brownian motion [Dys62]. A different proof of $(\Gamma^k(B))_k = \hat{B}_k$ via embedding interlacing Brownian motions into Gelfand-Tsetlin cone is given in [War07].

So we've introduced multidimensional discrete Pitman's theorem from RSK and path transform G^k, and continuous Pitman's theorem from transform Γ^k, what is missing is a continuous RSK. Although we no longer need it to connect Γ^k with Pitman, we can still define it and obtain a intertwining relationship between the kernel of input Brownian motion and output Dyson's non-colliding Brownian motion [O'C03a]. To construct it, note that the space dimension k is unchanged as discrete while the time dimension n has become continuous. Hence there's no immediate analogy of input matrix D and we'd rather define it taking input as continuous version of walk S, i.e. continuous functions $f_1, \ldots, f_k \in C([0,1])$ such that $f_i(0) = 0, \forall i$. The continuous RSK algorithm is defined in the spirit of the shape interpretation of output tableaux (1.6)(1.7). Since the analogy of shape L^j of jth subtableaux

of tableau P is

$$L^j \leftrightarrow \Gamma^j(f_j, f_{j-1}, \dots, f_1)^r;$$

The correspondence is defined as a mapping from $C([0,1])^k$ to $\mathrm{GT}^k \times C_0([0,1], \Omega)$ by

$$\begin{aligned}(f_1, \dots, f_k) \leftrightarrow (\tilde{P}, \tilde{Q});\ &\text{where}\\ \tilde{P} &= (\Gamma^1(f_1)(1)^r, \Gamma^2(f_2, f_1)(1)^r, \dots, \Gamma^k(f_k, f_{k-1}, \dots, f_1)(1)^r);\\ \tilde{Q} &= (\Gamma^k(f_k, f_{k-1}, \dots, f_1)(t)^r)_{0\le t\le 1}.\end{aligned}$$

5.2 Geometric Lifting

The transforms defined before are based on *max-plus algebra*. For example, given k-dimensional walks $(S_i)_{i\in[k]}$ the length of the first row of $\mathrm{sh}P(S)$ at time n are

$$G_k^k(S_k, S_{k-1}, \dots, S_1) = S_1 \triangledown \dots \triangledown S_k = \max_{\phi:(1,1)\to(n,k)} \sum_{(i,j)\in\phi} \xi_{ij}.$$

If we replace $(\vee, +)$ with $(+, \times)$, it turns to *conventional algebra*, under which RSK and Pitman's theorem are described in this section.

The discrete RSK in conventional algebra is called tropical RSK and has strikingly similar properties as the classical one. However, as is said earlier in Section 1.4, the alternative definition of classical RSK is derived from tropical RSK. Given a $n \times k$ input matrix D with positive real entries, we can define $\mathcal{L}$ in the same way as L as in (1.10)(1.11) except replacing "$\vee$" with "$+$" and "$+$" with "$\times$":

$$\begin{aligned}&\mathcal{T}_l(A) = \sum_{\substack{\pi_1:(1,1)\to(m,j-l+1),\pi_2:(1,2)\to(m,j-l+2),\dots,\pi_l:(1,l)\to(m,j)\\ \pi_1,\pi_2,\dots,\pi_l \text{ are non-intersecting}}} \prod_{(i,p)\in\pi_1\cup\dots\cup\pi_l} (A)_{i,p}, \quad A \in M(\mathbb{R}_+)_{m\times j};\\ &\mathcal{L}_1^j(m)\mathcal{L}_2^j(m)\dots\mathcal{L}_l^j(m) = \mathcal{T}_l(D^j(m)).\end{aligned} \tag{5.7}$$

And we can define the RSK correspodence in the same way as in (1.6)(1.7) by

$$\begin{aligned}\mathcal{P}(n) &= (\mathcal{L}^j(n))_{1\le j\le k};\\ \mathcal{Q}(n) &= (\mathcal{L}^k(m))_{1\le m\le n},\end{aligned}$$

such that for $i, j \in [k]$ the ith "row" of $\mathcal{P}$-tableaux is $\mathcal{L}_i(m) = (\mathcal{L}_i^i, \mathcal{L}_i^{i+1}, \dots, \mathcal{L}_i^k)(m)$ and the

shape of jth subtableau of $\mathcal{P}(m)$ is $\mathcal{L}^j(m) = (\mathcal{L}_1^j, \mathcal{L}_2^j, \ldots, \mathcal{L}_j^j)(m)$.

As an argument in [COSZ11], it should be noted that in the definition of $\mathcal{T}_l$ (5.7), $\mathcal{L}_i^j(m)$ makes sense only when $j \le k$ and $i \le j \wedge m$ (although a trivial exception is $\mathcal{L}_{m+1}^{m+1}(m) = 1$). Also the shape of $\mathcal{P}(n)$ and $\mathcal{Q}(n)$, i.e. $\mathcal{L}^k(n)$ are the same, the elements of $\mathcal{P}(n)$ and $\mathcal{Q}(n)$ that are "informative" are hence

$$(\mathcal{P}, \mathcal{Q})(n) \leftrightarrow (\mathcal{L}_i^j(m))_{(i,j,m)\in B_{n,k}},$$

where $B_{n,k}$ is as defined in (1.8). The set $B_{n,k}$ has exactly nk element. So we can define the RSK correspondence from set of $n \times k$ matrices onto itself [COSZ11] by $D \leftrightarrow \tilde{D}$, where $\tilde{D} = (\tilde{\xi}_{ij})$ with

$$\tilde{\xi}_{ij} = \begin{cases} \mathcal{L}_i^{i+j-1}(n) & 1 \le i \le k \wedge n, 1 \le j \le k-i+1; \\ \mathcal{L}_{k-j+1}^k(k+n-i-j+1) & 1 \le j \le k, k-j+1 \le i \le n. \end{cases}$$

Rather than computing the $\mathcal{L}$'s in the painful way as (5.7), there's also an algorithm to calculate them in a time-evolution way, like in the classical RSK correspondence. It is called *tropical* RSK algorithm and is constructed in [NY04], based on tropical RSK correspondence introduced in [Kir01]. The identity of the output of the algorithm and the $\mathcal{L}$'s can be found in [NY04, COSZ11].

A *geometric row insertion* inserts a vector into the $\mathcal{P}$-tableau in a similar way to row insertion into P-tableau in the classical setting. Given two l-vectors α and β, a row insertion inserting β into α is a function $r : (\mathbb{R}_+^l \times \mathbb{R}_+^l) \to (\mathbb{R}_+^l \times \mathbb{R}_+^{l-1})$ defined by

$$\begin{aligned} &r : (\alpha, \beta) \mapsto (\alpha', \beta') \text{ s.t.} \\ &\alpha_1' = \beta_1\alpha_1; \qquad \alpha_i' = \beta_i(\alpha_{i-1}' + \alpha_i); \qquad \beta_{i-1}' = \beta_i \frac{\alpha_i\alpha_{i-1}'}{\alpha_{i-1}\alpha_i'}, \quad 2 \le i \le l. \end{aligned}$$

Denote $\mathbf{e}_1^l = (1, 0, \ldots, 0)$ to be unit l-vector. the geometric row insertion can be extended to accept argument $\mathbf{e}_1^l$:

$$\begin{aligned} &r : (\alpha = \mathbf{e}_1^l, \beta) \mapsto (\alpha', \beta') \text{ s.t.} \\ &\alpha_1' = \beta_1\alpha_1; \qquad \alpha_i' = \beta_i(\alpha_{i-1}' + \alpha_i); \qquad \beta_{i-1}' = 1, \quad 2 \le i \le l. \end{aligned}$$

The second output β' in this case does not matter, as will be seen in the algorithm. Denote $\xi^m := (\xi_{m1}, \ldots, \xi_{mk})$ to be the mth row of input matrix. The tropical RSK algorithm is described as below.

1. Initialise $\mathcal{P}(0) = \mathcal{L}(0)$ to be empty, meaning a convention $\mathcal{L}_i(0) = e_1^{k+1-i}$, $i \in [k]$. Set $m = 1$.

2. If $m > n$ then we are done.

3. Set i=1.

4. Insert ξ^m into $\mathcal{L}_i(m-1)$ to update ξ^m and obtain $\mathcal{L}_i(m)$: $(\mathcal{L}_i(m), \xi^m) \leftarrow r(\mathcal{L}_i(m-1), \xi^m)$.

5. If $i < m \wedge k$ then $i \leftarrow i+1$ and go to step 4; otherwise $m \leftarrow m+1$ and go to step 2.

To consider tropical RSK with random input, let us first define Whittaker functions. For $\lambda \in \mathbb{R}^k$, let ψ_λ be the eigenfunction of

$$H := \Delta - 2\sum_{i=1}^{k-1} e^{x_{i+1}-x_i}$$

with eigenvalue $\|\lambda\|_2^2$ such that $(e^{-\mu(x)}\psi_\lambda(x))_x$ is bounded and $\lim_{x\to\infty, x\in\Omega} e^{-\mu(x)}\psi_\lambda(x) = 1$. ψ_λ are so called class-one Whittaker functions. For $y \in \mathbb{R}_+^k$ define function Ψ_λ by $\Psi_\lambda(y) = \psi_{-\lambda}(\log y_1, \log y_2, \ldots, \log y_k)$.

Recall in Chapter 2 in order to obtain a Markovian shape of output tableaux of classical RSK correspondence with "solvable" dynamics we set (ξ_{ij}) to be independently exponentially distributed with parameter $1 - p_i q_j$ such that $0 < p_i q_j < 1, \forall i \in [n], j \in [k]$.In the tropical RSK setting, to obtain a related evolution of the shape, (ξ_{ij}) also need to be independent random variables, but distributed according to inverse-gamma distribution with parameter $\tilde{p}_i + \tilde{q}_j$ such that $\tilde{p}_i + \tilde{q}_j > 0, \forall i \in [n], j \in [k]$. The inverse-gamma distribution with parameter θ has the density function $\frac{1}{\Gamma(\theta)} x^{-\theta-1} e^{-\frac{1}{x}}$.

This is worked in [COSZ11]. With such ξ_{ij}'s, the intertwining kernel from the shape $\mathcal{L}^k$ to the tableau $\mathcal{P}$ can be written down, so is the transition kernel of Markov chain $\mathcal{L}^k$. The push-forward law of the shape at time n is approached by the law of shape at time n with initial condition in $\mathbb{R}_+^k$. As an analogue of Schur measure (2.4), it is called *Whittaker measure* and written as

$$\mathbb{P}(\mathcal{L}^k(n) \in dy) = \left(\int_{\iota\mathbb{R}_+^k} s_k(\lambda) \prod_{m=1}^{n} \prod_{i=1}^{k} \frac{\Gamma(\tilde{q}_m + \lambda_i)}{\Gamma(\tilde{p}_i + \tilde{q}_m)} \Psi_{\tilde{p}}(y) \Psi_{-\lambda}(y) d\lambda \right) \frac{dy}{\prod_{j=1}^k y_j},$$

where $\iota = \sqrt{-1}$ is the unit imaginary number and $s_k(\lambda) = ((2\pi\iota)^k k! \prod_{i\neq j} \Gamma(\lambda_i - \lambda_j))^{-1}$.

Another fact revealing that inverse-gamma is the distribution corresponding to geometric distribution in classical RSK setting and tropical RSK is the RSK in conventional algebra

setting is as follows. Set a family of input matrices D^ϵ with entries ξ_{im}^ϵ being independently inverse-gamma distributed with parameter $\epsilon(\tilde{p}_i + \tilde{q}_m)$. The ϵ serves as "temperature" and the algebra turns to max-plus as $\epsilon \to 0$. Let D denote the input matrix whose entries ξ_{im} are independently exponentially distributed with parameter $\tilde{p}_i + \tilde{q}_m$. Then

$$\epsilon \log \xi_{im}^\epsilon \xrightarrow{\mathbf{d}} \xi_{im}, \text{ as } \epsilon \to 0.$$

Moreover, denote $\mathcal{L}(D^\epsilon)$ to be the output process of D^ϵ via tropical RSK and $L(D)$ to be the output of D via classical RSK. the remarkable fact is

$$\epsilon \log \mathcal{L}(D^\epsilon) \xrightarrow{\mathbf{d}} L(D)$$

as processes [COSZ11].

Now consider the continuous setting. The construction can be found in [O'C12]. As is the case in max-plus algebra continuous setting, we aim to build a path transform of input walk to obtain the output of RSK. The symmetric group $\mathcal{S}_k$ is generated by $s_1, \ldots, s_{k-1}$ where $s_i = (i, i+1)$ are adjacent transpositions. For any permutation $\sigma \in S_k$, it can be decomposed into a shortest product of these transpositions, called *reduced decomposition*

$$\sigma = s_{i_1} s_{i_2} \ldots s_{i_l}.$$

This is like a (not very efficient) sorting program, in which the worst case corresponds to the "longest" element of S_k. It is

$$\sigma_k = \begin{pmatrix} 1 & 2 & \ldots & k \\ k & k-1 & \ldots & 1 \end{pmatrix}$$

and one of its reduced decompositions is

$$\sigma_k = (s_1 s_2 \ldots s_{k-1})(s_1 \ldots s_{k-2}) \ldots (s_1 s_2)(s_1).$$

Now define operators $T_i : C[0,1] \to \mathbb{R}^k$ by

$$T_i f(t) = f(t) + \left(\log \int_0^t e^{f_{i+1}(s) - f_i(s)} ds \right) (\mathbf{e}_i - \mathbf{e}_{i+1}). \quad 1 \le i \le k-1. \tag{5.8}$$

And for any permutation σ with reduced decomposition $\sigma = s_{i_1} s_{i_2} \ldots s_{i_l} \in S_k$ define T_σ by

$$T_\sigma = T_{i_1} T_{i_2} \ldots T_{i_l}.$$

This is well-defined, i.e. T_σ does not depend on the choice of the reduced decomposition [BBO05]. The path transform on k-dimensional continuous functions $C([0,1])^k$ is defined as

$$\Pi^k := T_{\sigma_k}.$$

And the RSK correspondence is (to keep consistent with the previous definitions of output tableaux, we write it slightly different from what is in [O'C12].)

$$\begin{aligned} f &\leftrightarrow (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}});\\ \tilde{\mathcal{P}} &= (\Pi^1(f_1), \Pi^2(f_2, f_1), \ldots, \Pi^k(f_k, f_{k-1}, \ldots, f_1))(1);\\ \tilde{\mathcal{Q}} &= (\Pi^k(f_k, f_{k-1}, \ldots, f_1)(t))_{0\le t\le 1}. \end{aligned}$$

To see why, note by replacing $\log \int_0^t e$ in (5.8) with $\sup_{0\le s\le t}$ we obtain an alternative definition of Γ^k:

$$\begin{aligned} P_i f(t) &:= f(t) + \sup_{0\le s\le t}\{f_{i+1}(s) - f_i(s)\}(\mathbf{e}_i - \mathbf{e}_{i+1});\\ T_{s_{i_1}\ldots s_{i_l}} &:= T_{i_1}\ldots T_{i_l};\\ \Gamma^k f &= (P_{\sigma_k} f)^r. \end{aligned}$$

This definition first appears in [BBO05] among its generalisations to arbitrary finite Coxeter groups.

The path-transform Π^j, as expected, has the Greene-style property like (1.10)(5.7). For $l \le k$ denote by $(\phi^i)_{i\in[l]}$ arbitrary non-intersecting directed paths from $(0,1),(0,2),\ldots,(0,l)$ to $(1,k+1-l),(1,k+2-l),\ldots,(1,k)$ respectively. For each i, $\phi^i = (\phi^i_i,\ldots,\phi^i_{k+i-l})$ such that each ϕ^i_j is a path with fixed second coordinate from (t^i_{j-1}, j) to (t^i_j, j) and $0 = t^i_{i-1} \le t^i_1 \cdots \le t^i_{k+i-l-1} \le t^i_{k+i-l} = 1$. Denote $E(\phi^i) := \sum_{j=i}^{k+i-l} f_j(t^i_j) - f_j(t^i_{j-1})$. Then

$$((\Pi^k f)_1 + \cdots + (\Pi^k f)_l)(1) = \log \int \prod_{i=1}^{l} e^{E(\phi^i)} d\phi, \tag{5.9}$$

where the integral is with respect to the Lebesgue measure on the set of non-intersecting j-tuples up to time t.

Since Π^k is the path-transform in geometrical lifting context, we would expect a similar result to Theorem 5.7, that is, for B a k-dimensional Brownian motion, $\Pi^k(B)$ has the same law as B conditioned on some event. In the max-plus setting, the event is that the Brownian motion stays in Ω, this means $\sup_{t\ge 0}\{B_{i+1}(t) - B_i(t)\} = 0, i = 1,\ldots,k-1$. Substituting

$\sup_{t\geq 0}$ with $\log\int_0^\infty \exp$, the condition we expect would be concerned with

$$A^i_\infty := \int_0^\infty e^{B_{i+1}(s)-B_i(s)}, \quad i = 1, \dots, k-1. \tag{5.10}$$

The search for such a condition is brought up in [O'C04] and carried on in [BO11], in which it is anticipated that $\Pi^k B$ indeed has the same law as B conditioned on (A^i_∞) in an appropriate limiting sense.

On the other hand, the law of $\Pi^k B^\mu$ where B^μ is a k-dimensional Brownian motion with constant drift μ is given explicitly in [O'C12].

Theorem 5.8 ([O'C12])**.** *The process $\Pi^k B^\mu$ is a diffusion whose generator is $\frac{1}{2}\Delta + \nabla \log \psi_\mu \cdot \nabla$.*

When $k = 2$, this theorem is equivalent to "geometric" Pitman's theorem, where we consider

$$Z^\mu_t = \log \int_0^t e^{2B^\mu_s - B^\mu_t} ds = 2M^\mu_t - B^\mu_t,$$

where $2M^\mu_t := \log\int_0^t e^{2B^\mu_s} ds$. Then

Theorem 5.9 ([MY00])**.** *The process Z^μ is a Brownian motion with a drift, satisfying*

$$dZ^\mu_t = dW_t + \frac{d}{dx}\log K_\mu(e^{-x})\Big|_{x=Z^\mu_t} dt,$$

where K_μ is Macdonald function.

5.3 Continuum

We briefly construct the RSK correspondence in continuum setting. Only the geometric RSK is built in [OW11], in a similar way to the construction of tropical RSK from $\mathcal{T}_l$ (5.7). The space for the non-intersecting paths, formerly $[n] \times [k]$ in discrete case and $[0, 1] \times [k]$ in continuous case now changes to $[0, t] \times \mathbb{R}$. Moreoever, the randomness can infect space dimension now - for example we can consider a Brownian sheet and space-time white noise (see e.g. [Wal86]), denoted as W and $\dot{W}$ respectively. Recall in discrete setting, thinking of a Cartesian grid $[n] \times [k]$ with the same weights distribution as the input matrix, a directed path from $(1, 1)$ to (n, k) is the graph of a "random walk bridge" starting at 0 at time 0 and ending at $(k - n)/\sqrt{2}$ at time $(n + k - 2)/\sqrt{2}$ that is rotated by $\pi/4$ and then translated by $(1, 1)$. Therefore in the continuum setting we can consider a Brownian bridge starting

at 0 at time 0 and ending at x at time t as an analogue. After scaling, a fixed number of non-intersecting "random walk bridges" with neighbouring starting and ending points in the continuum setting are non-intersecting random walks with identical starting and ending points. Therefore heuristically, the counterpart of (5.9) in continuous setting and $\mathcal{T}_l$ in (5.7) in discrete setting should now be

$$\mathbb{E}\exp\left(\sum_{i=1}^{l}\int_0^t \dot{W}(s,X_s^i)ds\right),$$

where $(X^i)_{i\in[l]}$ are l non-intersecting Brownian bridges starting at 0 at time 0 and ending at x at time t. However due to the irregularity of white noise, the integrals in the bracket are distributions rather than functions, whose exponentials make no sense. So in [OW11] it is instead defined as

$$\begin{aligned}\mathsf{T}_l(t,x) &= p(t,x)^l\mathbb{E} : \exp : \left(\int_0^t \sum_{i=1}^{l}\dot{W}(s,X_s^i)ds\right)\\ &= p(t,x)^l\left(1+\sum_{k=1}^{\infty}\int_{\Delta_k(t)}\int_{\mathbb{R}^k} R_k^{(l)}((t_1,x_1),\dots,(t_k,x_k))W(dt_1,dx_1)\dots W(dt_k,dx_k)\right),\end{aligned}$$

where $p(t,x)=(2\pi t)^{-1}e^{-x^2/2t}$, $:\exp:$ is wick exponential, $\Delta_k(t)=\{0<t_1<\cdots<t_k<t\}$ and $R_k^{(l)}$ is the k-point correlation function for X. Like in (5.9) and (5.7), the components $(\mathsf{L}_i)_{i\geq 0}$ of the tableaux are defined recursively:

$$\mathsf{T}_l=\mathsf{L}_1\mathsf{L}_2\dots\mathsf{L}_l.$$

Then the RSK correspondence for Brownian sheet on $[0,t]\times\mathbb{R}$ are defined by

$$\begin{aligned}W|_{[0,t]\times\mathbb{R}} &\leftrightarrow (\mathsf{P},\mathsf{Q});\\ \mathsf{P} &= (\mathsf{L}_l(t,\cdot)|_{[0,\infty)}, l\geq 1);\\ \mathsf{Q} &= (\mathsf{L}_l(t,\cdot)|_{(-\infty,0]}, l\geq 1);\\ \mathrm{sh}\mathsf{P}=\mathrm{sh}\mathsf{Q} &= (\mathsf{L}_l(t,0), l\geq 1).\end{aligned}$$

For smooth potential $\phi\in C^\infty([0,t]\times\mathbb{R})$ of the environment, the RSK can be defined in the same way, except the partition functions can be directly defined as

$$p(t,x)^l\mathbb{E}\exp\left(\sum_{i=1}^{l}\int_0^t \phi(s,X_s^i)ds\right),$$

since ϕ is smooth.

Chapter 6

Applications

The results we have presented so far has many applications. In this chapter we mention some connections between them and random matrices, directed random polymers, corner growth models, simple exclusion processes, Kardar-Parisi-Zhang equation and universality class.

A k-dimensional *Gaussian Unitary Ensemble* (GUE) is a random matrix $A \in M(\mathbb{C})_{k\times k}$ defined as

$$(A)_{ij} = \begin{cases} \frac{1}{\sqrt{2}}(\zeta_{ij} + \iota\zeta_{ji}) & 1 \le i < j \le k; \\ \zeta_{ii} & 1 \le i = j \le k; \\ \frac{1}{\sqrt{2}}(\zeta_{ji} - \iota\zeta_{ij}) & 1 \le j < i \le k, \end{cases}$$

where $(\zeta_{ij})_{k\times k}$ are k^2 independent standard normal distributed random variables. A *Hermitian Brownian motion* (HBM) is defined in the same way with (ζ_{ij}) being independent Brownian motions. A (n,k)-*Laguerre Unitary Ensemble* (LUE) is defined as a random Hermitian matrix AA^*, where $A = X + iY \in M(\mathbb{C})_{k\times n}$ with $(X)_{ij} \overset{\mathbf{d}}{=} (Y)_{ij} \sim N(0, 1/2)$ being all independent. The first connection between RSK and random matrices, as was mentioned in Section 5.1 is: the k-dimensional non-colliding Brownian motion $\hat{B}$ has the same distribution as the eigenvalue processes of a k-dimensional HBM [Dys62].

Given an $n \times k$ matrix D, the directed path from $(1,1)$ to (n,k) with minimal sum, i.e. $G_1^k(S(D))$ describes the ground state energy of a *directed first passage percolation model*, which is the 0-temperature case of discrete directed random polymer. The free energy of the latter with temperater β^{-1} is $-\beta^{-1}\log\sum_{\phi:(1,1)\to(n,k)}\prod_{(i,j)\in\phi}\exp(-\beta\xi_{ij})$. On the other hand, the directed path from $(1,1)$ to (n,k) with maximal sum, i.e. $L_1^k(n)$ is related to the partition function of the *directed last percolation* model, which is again 0-temperature case of discrete directed random polymer. The partition function of continuous and continuum directed random polymer are also sometimes motivations for studying RSK correspondence (e.g. L_1

in Section 5.3) and path transforms (e.g. $\exp(\Pi_1^k(B_k, \dots, B_1)(1)))$, see [OW11, O'C12].

Thinking of the grid $\mathbb{N}_+^2$ for a directed last passage percolation model, we can associate it with a corner growth model. To do this, first define

$$A_L(m) = \{(n,k) \in \mathbb{N}_+^2 : L_1^k(n) + k + n - 1 \leq m\} + [-1,0]^2.$$

Then clearly $L_1^k(n) + k + n - 1 = m$ means $(n,k) \in A_L(m)$. By rotating $A_L(m)$ around the origin anti-clockwisely by $\pi/4$ and scaling (enlarging) it by $\sqrt{2}$, we obtain a corner growth model, which corresponds to a discrete-time simple exclusion process (SEP) on $\mathbb{Z}$ lattice with a wedge initial condition.

An *SEP* is a process $(\eta_t : t \geq 0)$ taking value in $\{-1,1\}$-sequence indexed by $\mathbb{Z}$. A *corner growth model* corresponding to an SEP considers the height function $h : [0,\infty) \times \mathbb{Z} \to \mathbb{Z}$, satisfying

$$h(0,0) = 0;$$
$$h(t,x) - h(t,x-1) = \eta_t(x).$$

SEP can be thought of as lattice sites $\mathbb{Z}$ occupied by particles, with $\eta_t(x) = 1$ ($\eta_t(x) = -1$) indicates at time t site x is occupied (unoccupied) by a particle. At a time a particle can jump to the left or right adjacent unoccupied site, or stay where it is. There are several types of initial conditions of interest [Cor11]. One of them is called *wedge*, defined as $h(0,x) = |x|$ and $\eta_0(x) = \mathbb{I}_{x>0}$. As mentioned before this corresponds to the discrete directed last passage percolation. In the our discrete-time model, if we take $p = 1$ and $q_j = q$ to be constant, then each time particles independently jump to the left unoccupied site with probability $1-q$ and stays on its site with probability q. It turns out in this case the fluctuation of the growth model follows Tracy-Widom distribution, i.e. the distribution of the largest eigenvalue of the GUE [Joh00].

In the continuous-time case, one class of SEP of interest is by making particles jump exponentially with rate q to the left and p to the right unoccupied site, such that $p + q = 1$. Denote $\gamma = q - p$, then $\gamma = 1$, $\gamma \neq 0$ and $\gamma = 0$ corresponds to *totally asymmetric* (TASEP), *asymmetric* (ASEP) and *symmetric* (SSEP) simple exclusion processes. Specifically, TASEP can be achieved by making all the ξ's distributed exponentially with parameter 1. In this case, by an simple argument of scaling, the distribution of directed last passage percolation of grid $[n] \times [k]$ with $n \geq k$ is the same of that of the largest eigenvalue of (n,k)-LUE [Joh00]. For general parameters, [DW09] proves a generalised result in a process (indexed by n) level. However, going back to parameter 1 case, when considering asymptotic fluctuation, the distribution is again scaled into Tracy-Widom law [Joh00].

In both of the above cases, the asymptotics of the distributions differ in two aspects from the normal central limit theorem where the fluctuation is Gaussian. First, the standard deviation is a third of the order of mean, rather than half, which reflects the nonlinear relation between the statistics and the number. The second is that the attractor is not Gaussian. Actually this kind of fluctuation happens in other models as well. For example, in [BDJ99], it is proved that the $l(\sigma)$ for $\sigma \in \mathcal{S}_N$ be an uniformly picked permutation scaled into GUE statistics with the same $1/3$ exponential of standard variance. This kind of central limit theorems is called KPZ (Kardar-Parisi-Zhang) universality, as opposed to Gaussian universality in the usual cases. It gets its name from the KPZ equation, a stochastic PDE:

$$\partial_t H = \frac{1}{2}\partial_{xx} H + \frac{1}{2}(\partial_x H)^2 + \dot{W},$$

where $\dot{W}$ is a space-time white noise. This equation is ill posed since $(\partial_x H)^2$ makes no sense. However for a solution of well-posed stochastic heat equation (SHE)

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + Z\dot{W},$$

by denoting $H = \log Z$ and putting H into the SHE, it turns out H formally satisfies the KPZ equation. Such an H is called a Hopf-Cole solution. It is the scaling limit of height function h of some SEP model. On the other hand, the solution Z to SHE with initial condition $Z(0, x) = \delta(x)$ (corresponding to "narrow wedge" initial condition of KPZ equation) is L_1 defined in Section 5.3, see [OW11].

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